Gerber-Shiu Function in a Discrete-time Risk Model with Dividend Strategy

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Authors’ contributions

This work was carried out in collaboration between both authors. Both authors have equal contributions. Both authors have read and approved the final manuscript.

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Abstract

In this paper, a discrete-time risk model with dividend strategy and a general premium rate is considered. Under such a strategy, once the insurer’s surplus hits a constant dividend barrier \( b \), dividends are paid off to shareholders at \( \alpha \) instantly. Using the roots of a generalization of Lundberg’s fundamental equation and the general theory on difference equations, two difference equations for the Gerber-Shiu discounted penalty function are derived and solved. The analytic results obtained are utilized to derive the probability of ultimate ruin when the claim sizes is a mixture of two geometric distributions. Numerical examples are also given to illustrate the applicability of the results obtained.

Keywords: Compound binomial model; two-step premium; defective renewal equation; Gerber-Shiu discounted penalty function; dividend strategy.

1 Introduction

Risk theory has a long development time, Lundberg [1] and Gramer [2] established the connection of risk theory. The compound binomial model that was first proposed by Gerber [3] have received considerable attention. For
instance, Shiu [4], Willmot [5] and Dickson [6] have analyzed the compound binomial model. Markov chain is understood to be a stochastic process in discrete time possessing a certain conditional independence property. The state space may be finite, countably infinite or even more general. Cossette et al. [7] consider the so-called compound Markov binomial model which introduces dependency between claim occurrences. For an generalization of the classical risk model see Landriault [8]. Furthermore, in the discrete time risk model, the issue related to dividend is also widely considered.

Dividend strategies for insurance risk models were first proposed by DeFinetti [9] to reflect more realistically the surplus cash flows in an insurance portfolio. Because of the certainty of ruin for a risk model with a constant dividend barrier, the calculation of the Gerber-Shiu discounted penalty function is a major problem of interest in the context. Among the class of discrete-time risk models, Tan and Yang [10] derived a recursive algorithm to compute a particular class of Gerber-Shiu penalty functions in the framework of the compound binomial model with randomized dividend payments. Landriault [11] then generalized Tan and Yang’s model to consider the compound binomial model with a multi-threshold dividend structure and randomized dividend payments. In the discrete time risk model, He and Yang [12] considered that dividends are paid randomly to shareholders and policyholders in the framework of the compound binomial model. In the framework of a discrete semi-Markov risk model, a randomized dividend policy is studied by Yuen et al. [13]. Zhang and Liu [14] consider a discrete-time risk model with a mathematically tractable dependence structure between interclaim times and claim sizes in the presence of an impulsive dividend strategy.

The paper is structured as follows: a brief description of the discrete-time model and the introduction of the Gerber-Shiu discounted penalty function are considered in Section 2. In section 3, we obtain and solve a non-homogeneous difference equation satisfied by the the Gerber-Shiu discounted penalty function \( m_{\alpha}(c,b) \). Closed-form solutions for \( m_{\alpha}(u) \) are obtained when the claim sizes is a mixture of two geometric distributions and corresponding numerical examples are also provided in Section 4.

2 The model

Throughout, denote by \( N \) the set of natural numbers and \( N^+ = N \cup \{0\} \). In the compound binomial model, the claim number process \( \{N_k, k \in N\} \) is assumed to be a renewal process with independent and identically distributed (i.i.d.) interclaim times \( \{W_j, j \in N^+\} \) having probability mass function (p.m.f.) \( f_{W}(j) = q(1-q)^{j-1} \) \( j \in N^+ \). Equivalently, the probability of having a claim is \( p(0 < p < 1) \) and the probability of no claim is \( q = 1 - p \). The individual claim amount r.v.’s(random variables) \( \{X_j, j \in N^+\} \) form a sequence of strictly positive, integer-valued and i.i.d. r.v.’s. We suppose that the r.v.’s \( \{X_j, j \in N^+\} \) are distributed as a generic r.v. \( X \) with p.m.f. \( f(x) \), probability generating function (p.g.f.) \( \tilde{f}(x) \). Moreover, it is assumed that the r.v.’s \( W_{1}, W_{2}, \ldots \) and \( X_{1}, X_{2}, \ldots \) are mutually independent. Let \( S_k = \sum_{i=1}^{N_k} X_i \) be the total amount of settled claims up the end of the kth time period with \( S_0 = 0 \).

Suppose that premiums are received at the beginning of each time period, and claims are paid out at the end of each time period. Denote \( u \geq 0 \) to be the initial surplus, \( b > 0 \) the constant barrier level, and \( c_1 > 0 \) the annual premium. Under such a strategy, let \( \alpha(0 < \alpha \leq c_1) \) be the annual dividend rate, once the insurer’s surplus at time \( k \) hits or exceeds a constant dividend barrier \( b \), dividends are paid off to shareholders at \( \alpha \) instantly. In this case, the net premium after dividend payments is \( c_2 = c_1 - \alpha \geq 0 \). The corresponding surplus of the insurer at the end of the kth time period is \( U_k(\alpha) \) for \( k = 1, 2, \ldots \) can be described as
\[ U_b(k) = \begin{cases} U_b(k-1) + c_1 - \eta_k X_k, & U_b(k-1) \leq b \\ U_b(k-1) + c_2 - \eta_k X_k, & U_b(k-1) > b \end{cases}, \]  

(1)

where \( U_b(0) = u \). \{\eta_k \in N\} is an independent and identically distributed Bernoulli sequence, we denote by \( \eta_k = 1 \) the event of having a claim at the time \( k \) and denote by \( \eta_k = 0 \) the event that no claim at the time \( k \). We assume that \( P(\eta_k = 1) = p \) and \( P(\eta_k = 0) = 1 - p = q \) and surplus process \( U_b(k) \) has a positive drift by letting \( c_2 > pE[X] \) (known as the positive security loading condition in ruin theory).

Define \( \tau_b = \min\{k : U_b(k) < 0\} \) to be the time of ultimate ruin. Let \( v \) be a constant annual discount rate for each period. When ruin occurs, \( U_b(\tau - 1) \) is the surplus one period prior to ruin and \( |U_b(\tau)| \) is the deficit at ruin. For \( v \in (0,1] \), the well-known Gerber-Shiu discounted penalty function is then defined as

\[ m(u; b) = E\left[ v^{\tau_b} \omega(U_{\tau_b-1}, U_{\tau_b}, I_{\tau_b < \infty}) \bigg| U_0 = u \right], \]  

(2)

where \( \omega : N \times N^+ \rightarrow R \) is a penalty function and \( I_{\{Q\}} \) is the indicator function of an event \( Q \). Also, we consider some special cases of (2) with successively simplified the penalty functions. If \( \omega(n_1, n_2) = 1 \) for \( (n_1, n_2) \in N \times N^+ \), we get the generating function of the time to ruin, i.e.

\[ m_b(u) = E\left[ v^{\tau_b} I_{\{\tau_b < \infty\}} \bigg| U_0 = u \right]. \]

### 3. The Gerber-Shiu discounted penalty function

In this section, we derive two difference equations for the Gerber-Shiu discounted penalty function: one for the initial surplus below the barrier level \( b \) and the other for the initial surplus above the barrier level \( b \). Clearly, the Gerber-Shiu discounted penalty function \( m(u; b) \) behaves differently, depending on whether its initial surplus \( u \) is below or above the barrier level \( b \). Hence, we write

\[ m(u; b) = \begin{cases} m_1(u), & 0 \leq u < b \\ m_2(u), & u \geq b \end{cases}. \]

In order to identify the structural form of the solution for the Gerber-Shiu discounted penalty function, three cases will be considered separately.

#### 3.1 For initial surpluses less than the barrier \( b \)

In the first scenario, the initial surplus below the barrier \( b \), for \( u = 0,1,\ldots,b-c_1-1 \), we have

\[ m_1(u) = vq m_1(u + c_1) + vp \sum_{j=1}^{u+c_1} m_1(u + c_1 - j) f(j) + vp \sum_{j=u+c_1+1}^{\infty} \omega(u + c_1; j - u - c_1) f(j) \]

\[ = vq m_1(u + c_1) + vp\left[m_1 * f\right](u + c_1) + \gamma_1(u), \]

where
\[ \gamma_1(u) = v p \sum_{j=0}^{\infty} \delta(u + c_1; j - u - c_1)f(j). \]

and \( m_1 \ast f \) holds for the convolution product of \( m_1 \) and \( f \).

To state that (3) is a non-homogeneous difference equation of order \( c_1 \), we re-express (3) according to the forward difference operator \( \Delta \) and its property (see Chapter 2 of Kelly & Peterson [15]),

\[ m(u + c) = \sum_{j=0}^{c_1} \binom{c_1}{j} \Delta^j m(u), \quad (4) \]

substituting (4) into (3) shows

\[ m_1(u) = v q \sum_{j=0}^{c_1} \binom{c_1}{j} \Delta^j m_1(u) + v p \sum_{j=0}^{c_1} \binom{c_1}{j} \Delta^j (m_1 \ast f)(u) + \gamma_1(u), \quad (5) \]

for \( u = 0, 1, ..., b - c_1 - 1 \). (5) can be simplified to

\[ \sum_{j=0}^{c_1} a_{1,j} \Delta^j m_1(u) = \sum_{j=0}^{c_1} b_{1,j} \Delta^j (m_1 \ast f)(u) + \gamma_1(u), \quad (6) \]

where

\[ a_{1,j} = \left[ I_{j=0} - v q \binom{c_1}{j} \right], \quad b_{1,j} = v p \binom{c_1}{j}. \]

and \( A_i(z), B_i(z) \) are polynomials (in \( z \)) defined as

\[ A_i(z) = \sum_{j=0}^{c_1} a_{1,j} z^j, B_i(z) = \sum_{j=0}^{c_1} b_{1,j} z^j. \]

becomes

\[ A_i(\Delta) m_1(u) = B_i(\Delta)(m_1 \ast f)(u) + \gamma_1(u), \quad u = 0, 1, ..., b - c_1 - 1. \quad (7) \]

We know from (7) that \( m_1(u) \) satisfies a non-homogeneous difference equation of order \( c_1 \). From the general theory on difference equations, every solution to a \( c_1 \)-th order difference equation can be expressed as a particular solution to this difference equation plus a linear combination of \( c_1 \) linearly independent solutions to the associated homogeneous difference equation (cf. Elaydi [16], Theorem 2.30). Therefore, for \( u = 0, 1, ..., b - 1 \), the Gerber-Shiu discounted penalty function can be expressed as

\[ m_1(u) = \phi_1(u) + \sum_{j=0}^{c_1-1} \alpha_{1,j} \gamma_{1,j}(u), \quad u = 0, 1, ..., b - 1. \quad (8) \]
where \( \{ y_{1,j}(u) \}_{u=0}^{\infty} \) \((j = 0, 1, ..., c_1 - 1)\) are \(c_1\) fundamental solutions to the following homogeneous difference equation

\[
A_1(\Delta) y_1(u) = B_1(\Delta)(y_1 * f)(u) \quad u \geq 0. \tag{9}
\]

\( \{ \phi_i(u) \}_{u=0}^{\infty} \) is a particular solution to

\[
A_1(\Delta) \phi_i(u) = B_1(\Delta)(\phi * f)(u) + \gamma_i(u) \quad u \geq 0. \tag{10}
\]

Combining (3) and (9), we get

\[
y_1(u) = vq y_1(u + c_1) + vp(y_1 * f)(u + c_1). \tag{11}
\]

Multiplying (11) by \( z^{u+c_1} \) and then summing over \( u \) from 0 to \( \infty \) lead to

\[
\sum_{u=0}^{\infty} z^{u+c_1} y_1(u) = vq \sum_{u=0}^{\infty} z^{u+c_1} y_1(u + c_1) + vp \sum_{u=0}^{\infty} z^{u+c_1} (y_1 * f)(u + c_1), \tag{12}
\]

routine calculations lead to

\[
z^\gamma \tilde{y}_1(z) = vq \left[ \tilde{y}_1(z) - \sum_{u=0}^{c_1-1} z^u y_1(u) \right] + vp \left[ \tilde{y}_1(z) \tilde{f}(z) - \sum_{u=0}^{c_1-1} z^u (y_1 * f)(u) \right].
\]

After some algebra, one could see that (12) can be written as

\[
\tilde{y}_1(z) = \frac{-vq \sum_{u=0}^{c_1-1} z^u y_1(u) + p \sum_{u=j+1}^{c_1-1} z^u (y_1 * f)(u)}{z^\gamma - vq - vp \tilde{f}(z)}. \tag{13}
\]

By choosing \( y_{1,j}(u) = I_{\{j,u\}} \) for \( j,u \in \{0, 1, ..., c_1 - 1\} \). According to (13), the generating function associated to the fundamental solution \( \{ y_{1,j}(u) \}_{u=0}^{\infty} \) is

\[
\tilde{y}_{1,j}(z) = \frac{-vq \left( qz^j + p \sum_{u=j+1}^{c_1-1} z^u f(u-j) \right)}{z^\gamma - vq - vp \tilde{f}(z)} = \frac{-R_{1,j}(z)}{h_{1,j}(z) - h_{1,2}(z)} \quad u \geq 0, \tag{14}
\]

where

\[
\tilde{h}_{1,1}(z) = z^\gamma, \quad \tilde{h}_{1,2}(z) = vq + vp \tilde{f}(z), \quad R_{1,j}(z) = v \left( qz^j + p \sum_{u=j+1}^{c_1-1} z^u f(u-j) \right).
\]

**Lemma 3.1** : When \( v \in (0,1) \), the denominator in (14) has exactly \( c_1 \) zeros, say \( \{ z_i \}_{i=1}^{c_1} \) inside the unit circle \( C = \{ z : |z| = 1 \} \).
Lemma 3.2: When \( v = 1 \), the denominator in (14) has exactly \( c_1 - 1 \) zeros, say \( \{z_j \}_{j=1}^{c_1-1} \) inside the unit circle \( C = \{ z : |z| = 1 \} \) and another trivial root \( z_{c_1} = 1 \).

For the rest of the paper, we assume that all \( \{z_j \}_{j=1}^{c_1-1} \) are distinct, since the analysis of the multiple roots of Lundberg’s generalized equation leads to tedious derivations.

Let \( \pi_i(z) = \prod_{j=1}^{c} (z - z_j) \) and \( \pi'_i(z_k) = \prod_{j \neq k} (z_k - z_j) \), from Liu and Bao [17], we have

\[
\frac{\widetilde{h}_{i,1}(z) - \widetilde{h}_{i,2}(z)}{\pi_i(z)} = 1 - vP_T T_{z_1} \ldots T_{z_i} T_{z_1} f(c_1),
\]

(15)

where \( T_z \) is an operator (see Li [18]) defined as

\[
T_z y(c) = \sum_{u=0}^{\infty} z^u y(u + c) = \sum_{u=c}^{\infty} z^{u-c} y(u).
\]

(14) can be rewrote as

\[
\widetilde{y}_{i,j}(z) = \frac{-R_{i,j}(z)}{\pi_i(z)} = \frac{\pi_i(z)}{h_{i,1}(z) - h_{i,2}(z)}.
\]

(16)

Regarding the numerator in (16), partial fractions yield the equivalent representation

\[
\frac{-R_{i,j}(z)}{\pi_i(z)} = \sum_{k=1}^{c_1} \frac{R_{i,j}(z_k)}{\pi_i(z_k)} \frac{1}{z_k - z}.
\]

(17)

By inserting (15) and (17) into (16), we obtain

\[
\widetilde{y}_{i,j}(z) = v \sum_{k=1}^{c_1} \frac{R_{i,j}(z_k)}{\pi_i(z_k)} \frac{1}{z_k - z}.
\]

(18)

Theorem 3.1: For \( j = 0, 1, \ldots, c_1 - 1 \), \( y_{i,j}(u) \) satisfies the following defective renewal equation

\[
y_{i,j}(u) = \zeta_1 \sum_{n=0}^{u} y_{i,j}(u - n) \chi_1(n) + \zeta_1'(u),
\]

(19)

where

\[
\zeta_1 = vP_T T_{z_1} \ldots T_{z_{c_1}} T_{z_1} f(c_1), \quad \chi_1(n) = \frac{T_{z_1} \ldots T_{z_{c_1}} T_{z_1} f(c_1 + n)}{T_{z_1} \ldots T_{z_{c_1}} T_{z_1} f(c_1)}, \quad \zeta_1'(u) = \sum_{k=1}^{c_1} \frac{R_{i,j}(z_k)}{\pi_i(z_k)} \left( \frac{1}{z_k} \right)^{u+1}.
\]
Now, we turn our attention to the calculation of the particular solutions \( \{ \phi_i(u) \}_{i=0}^{\infty} \), combining (3) and (10), \( \{ \phi_i(u) \}_{i=0}^{\infty} \) satisfies

\[
\phi_i(u) = vq \phi_i(u + c_i) + vp(\phi_1(f)(u + c_i) + \gamma_1(u)).
\]  

We use a solution procedure analogous to the fundamental solutions, we get

\[
\tilde{\phi}_i(z) = \frac{z^{c_i} \tilde{\gamma}_1(z) - q \sum_{u=0}^{c_i-1} z^u \phi_i(u) + p \sum_{u=0}^{c_i-1} z^u (\phi_1(f)(u))}{z^{c_i} - vr - vp \tilde{f}(z)}
\]  

where \( \tilde{\gamma}_1(z) = v \sum_{u=0}^{c_i-1} z^u \phi_i(u) + p \sum_{u=0}^{c_i-1} z^u (\phi_1(f)(u)) \) is a polynomial of degree \( c_i - 1 \) (or less) in \( z \). It is known from (35) in Liu and Zhang [14] that

\[
\frac{z^{c_i} T_{c_0} \tilde{\gamma}_1(0) - Q_{i,j}(z)}{z^{c_i} - vr - vp \tilde{f}(z)} = T_{c_{z_1}} \ldots T_{c_{z_l}} T_{c_i} \tilde{\gamma}_1(0).
\]  

By substituting (15) and (22) into (21), we get

\[
\tilde{\phi}_i(z) = \frac{T_{c_{z_1}} \ldots T_{c_{z_l}} T_{c_i} \tilde{\gamma}_1(0)}{1 - vp T_{c_{z_1}} \ldots T_{c_{z_l}} f(c_i)} = \frac{\tilde{\phi}_i(z)}{1 - vp T_{c_{z_{j_1}}} \ldots T_{c_{z_{j_q}}} f(c_i)}.
\]  

**Theorem 3.2:** For \( u \in N \), it holds that

\[
\phi_i(u) = z_i \sum_{n=0}^{u} \phi_i(u - n) \chi_i(n) + \tilde{\phi}_i(u).
\]  

where \( \tilde{\phi}_i(u) = T_{c_{z_1}} \ldots T_{c_{z_l}} T_{c_i} \gamma_1(u) \).

In the second scenario, for \( u = b - c_i, \ldots, b - 1 \),

\[
m_1(u) = vq m_2(u + c_i) + vp(m* f)(u + c_i) + \gamma_1(u).
\]  

**3.2 For initial surpluses equal to or more than the barrier \( b \)**

The last scenario, for \( u \geq b \),

\[
m_2(u) = vq m_2(u + c_2) + vp(m* f)(u + c_2) + \gamma_2(u).
\]
where \( \gamma_2(u) = vp \sum_{j=0}^{\infty} \omega(u + c_2; j - u - c_2)f(j). \)

The structural form (8) for \( m_1(u) \) is expressed in terms of the \( \alpha_{1,j} \), and also depends on \( m_2(u) \) in (26). In order to drive the solutions of \( m(u) \), shifting the argument \( u \) in (26) by \( b \) units, for \( u \geq 0 \) (26) can be rewritten as

\[
m_2(u + b) = vqm_2(u + b + c_2) + vp \sum_{j=b}^{u+b+c_2} m_2(j)f(u + b + c_2 - j)
+ vp \sum_{j=0}^{b-1} m_1(j)f(u + b + c_2 - j) + \gamma_2(u + b),
\]

Let \( \xi_2(u) \equiv m_2(u + b) \), (26) becomes

\[
\xi_2(u) = vq \xi_2(u + c_2) + vp(\xi_2 * f)(u + c_2) + \eta(u), \tag{27}
\]

where \( \eta(u) = vp \sum_{j=0}^{b-1} m_1(j)f(u + b + c_2 - j) + \gamma_2(u + b) \).

We use a solution procedure analogous to that of Section 3.1. \( \xi_2(u) \) satisfies

\[
A_2(\Delta) \xi_2(u) = B_2(\Delta)(\xi_2 * f)(u) + \eta(u) \quad u \geq 0, \tag{28}
\]

where

\[
A_2(z) = \sum_{j=0}^{c_2} a_{2,j} z^j, B_2(z) = \sum_{j=0}^{c_2} b_{2,j} z^j, \quad a_{2,j} = I_{(j=0)} - vq \left( \frac{c_2}{j} \right), b_{2,j} = vp \left( \frac{c_2}{j} \right).
\]

From the general theory on difference equations, can be expressed as

\[
m_2(u + b) = \phi_2(u) \quad u = 0, 1, \ldots
\]

where \( \{ \phi_2(u) \}_{u=0}^{\infty} \) satisfies

\[
A_2(\Delta) \phi_2(u) = B_2(\Delta)(\phi_2 * f)(u) + \eta(u) \quad u \geq 0. \tag{29}
\]

Some solution procedures are omitted, similar discussions can be find in Section 3.1. Generating function of the particular solution \( \phi_2(u) \) is

\[
\tilde{\phi}_2(z) = \frac{z^{c_2} \tilde{\eta}(z) - v^q \sum_{u=0}^{c_2-1} z^u \phi_2(u) + p \sum_{u=0}^{c_2-1} z^u (\phi_2 * f)(u)}{z^{c_2} - vq - vp \tilde{f}(z)}
= \frac{z^{c_2} T \eta(0) - R_{2,j}(z)}{h_{2,j}(z) - \tilde{h}_{2,j}(z)}, \tag{30}
\]

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Where
\[ \tilde{h}_{2,j}(z) = z^{c_i}, \quad \tilde{h}_{2,j}(z) = vq + v p \tilde{f}(z), \quad R_{2,j}(z) = v \left\{ q \sum_{n=0}^{z-1} z^n \phi_2(u) + p \sum_{n=0}^{z-1} z^n \left( \phi_2 * f \right)(u) \right\}. \]

**Theorem 3.3**: For \( u \in N \), it holds that
\[ \phi_2(u) = \zeta_2 \sum_{n=0}^{u} \phi_2(u-n)X_2(n) + \vartheta_2(u), \quad (31) \]

where
\[ \zeta_2 = vpT_1T_{z_2} \ldots T_{z_i} f(c_2), \quad X_2(n) = \frac{T_{z_2} \ldots T_i f(c_2 + n)}{T_{z_2} \ldots T_i f(c_2)}, \quad \vartheta_2(u) = T_{z_2} \ldots T_i \eta(u). \]

So for \( u \geq b \),
\[ m_2(u) = \xi_2(u-b) = \phi_2(u-b). \quad (32) \]

**4 Numerical results**

It is well-known that \( f(x) = (1-\rho)x^0 \) is a geometric distribution. In this section, it is further assumed that \( f(x) \) is a mixture of two geometric distributions with \( f(x) = (1-\rho_1)x^0 + (1-\theta)(1-\rho_2)x^{\rho_2} \).

Obviously, probability generating function is \( \tilde{g}(x) = \frac{z((1-\rho_1)(1-\rho_2) + \beta(1-z))}{(1-\rho_1)(1-\rho_2)} \) where \( \beta = \theta \rho_2 (1-\rho_1) + (1-\theta) \rho_1 (1-\rho_2), \) and mean is \( \mu = \frac{\theta}{1-\rho_1} + \frac{1-\theta}{1-\rho_2} \). We rewrite (16) as
\[ \tilde{y}_{1,j}(z) = \frac{-R_{1,j}(z)(1-\rho_1 z)(1-\rho_2 z)}{\Lambda_1(z)}. \quad (33) \]

where
\[ \Lambda_i(z) = z^c(1-\rho_1 z)(1-\rho_2 z) - vq(1-\rho_1 z)(1-\rho_2 z) - vp[z(1-\rho_1)(1-\rho_2) + \beta(1-z)](i = 1, 2) \]

Since \( \Lambda_i(z) \) is a polynomial of degree \( c_i + 1 \), with leading coefficient \( \rho_1 \rho_2 \), it can be expressed as
\[ \Lambda_i(z) = \rho_1 \rho_2 \pi_i(z) \prod_{j=1}^{h} (z-\xi_j), \]

where \( \xi_j \) are solutions of \( \Lambda_i(z) \) on the complex plane. It is notable that \( \xi_j \) have a module larger than 1, from performing partial fraction, we have

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\[
\frac{\pi_i(z)(1-\rho_i z)(1-\rho_z)}{\Lambda_i(z)} = \frac{\pi_i(z)(1-\rho_i z)(1-\rho_z)}{\rho_i \rho_2 \pi_i(z) \prod_{j=1}^{h} (z-\xi_j)} = 1 + \sum_{i=1}^{h} \frac{\omega_i}{\xi_i - z},
\]  

(34)

where

\[
\omega_i = \frac{\prod_{k=1}^{h} (\rho_k^{-1} - \xi_i)}{\prod_{k=1, k \neq i} (\xi_k - \xi_i)}.
\]

For \(i = 1\), substituting (34) into (16) shows

\[
\tilde{y}_{1,j}(z) = \frac{(1-\rho_1 z)(1-\rho_z)}{\Lambda_1(z)} \sum_{k=1}^{c_1} \frac{\tilde{R}_{1,j}(z_k)}{\pi_1(z_k)} \frac{\pi_1(z)}{z_k - z} = \sum_{k=1}^{c_1} \frac{\tilde{R}_{1,j}(z_k)}{\pi_1(z_k)} \left( 1 + \sum_{i=1}^{h} \frac{\omega_i}{\xi_i - z} \right) \frac{1}{z_k - z}.
\]

(35)

Upon inversion, we obtain from (35) that

\[
y_{1,j}(u) = \sum_{k=1}^{c_1} \frac{\tilde{R}_{1,j}(z_k)}{\pi_1(z_k)} \left[ z_k^{-(u+1)} + \sum_{i=1}^{h} \omega_i \sum_{l=0}^{h} z_k^{-(u+1-l)} z_l^{-(l+i)} \right] = \sum_{k=1}^{c_1} \frac{\tilde{R}_{1,j}(z_k)}{\pi_1(z_k)} \left( 1 - \sum_{i=1}^{h} \frac{\omega_i}{z_k - \xi_i} \right) z_k^{-(u+1)} + \sum_{k=1}^{c_1} \frac{\tilde{R}_{1,j}(z_k)}{\pi_1(z_k)} \sum_{i=1}^{h} \omega_i z_k^{-(u+1+i)}. \]

(36)

Use the same method, we obtain from (30) and (34) that,

\[
\tilde{\phi}(z) = \frac{\pi_1(z)(1-\rho_1 z)(1-\rho_z)}{\Lambda_1(z)} \tilde{\eta}(z) = \left( 1 + \sum_{j=1}^{h} \frac{\omega_j}{\xi_j - z} \right) \tilde{\eta}(z),
\]

(37)

Upon the inversion of the generating functions, one obtains from (37) that

\[
\phi(u) = \tilde{\eta}(u) + \sum_{j=1}^{h} \omega_j \sum_{l=0}^{u} z_j^{-(u+1-l)} \tilde{\eta}(l).
\]

(38)

**Example:** Suppose \(c_1 = 2, c_2 = 1, p = 0.2, q = 0.8, v = 0.95, \rho_1 = 0.3, \rho_2 = 0.6\). from (33)

\[
\Lambda_1(z) = z^2(1-\rho_1 z)(1-\rho_z) - vq(1-\rho_1 z)(1-\rho_z) - v\beta[z(1-\rho_1)(1-\rho_z) + \beta(1-z)],
\]

\[
\Lambda_2(z) = z(1-\rho_1 z)(1-\rho_z) - vq(1-\rho_1 z)(1-\rho_z) - v\beta[z(1-\rho_1)(1-\rho_z) + \beta(1-z)].
\]

And the relatively safety loading condition \(c_2 - p \mu > 0\) holds for all \(\theta \in (0,1)\). Hence, \(\theta\) is chosen to be 0.1, 0.3, 0.5, 0.7, 0.9, respectively. By solving Lundberg’s equation \(\Lambda_1(z) = 0\), we obtain the values of \(z_j\)’s and \(\xi_j\)’s, see Table 1. By solving Lundberg’s equation \(\Lambda_2(z) = 0\), we obtain the values of \(z_k\)’s and \(\xi_k\)’s, see Table 2.
Explicit expressions for \( y_{i,j}(u) \) is determined by (36), so we obtain the values of \( y_{i,j}(u) \) for \( \theta = 0.5, c_1 = 2, c_2 = 1, p = 0.2, q = 0.8, \rho_1 = 0.3, \rho_2 = 0.6, v = 0.95, b = 10 \).

For instance, one has for \( \theta = 0.5 \),

\[
y_0(u) = -0.42464 \times (-0.88633)^u + 0.47519 \times 0.96901^u - 0.05183 \times 1.56583^u + 0.00129 \times 3.40325^u
\]

\[
y_1(u) = 0.42071 \times (-0.88633)^u + 0.55572 \times 0.96901^u - 0.04023 \times 1.56583^u + 0.00056 \times 3.40325^u
\]

Then solve a system of linear equations with \( a_{i,j} \), Table 3 lists the values of \( a_{i,j} \)’s.

### Table 3. Numerical results of \( a_{i,j} \) for \( b = 10 \).

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>0.1</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_{1,0} )</td>
<td>0.001602</td>
<td>0.00103</td>
<td>0.00059</td>
<td>0.00027</td>
<td>5.109949 \times 10^{-5}</td>
</tr>
<tr>
<td>( a_{1,1} )</td>
<td>0.001703</td>
<td>0.00109</td>
<td>0.00063</td>
<td>0.00029</td>
<td>6.318151 \times 10^{-5}</td>
</tr>
</tbody>
</table>

Explicit expressions for \( \phi_j(u) \) is determined by (38) so we get the values of \( \phi_j(u) \) for \( \theta = 0.5, c_1 = 2, c_2 = 1, p = 0.2, q = 0.8, \rho_1 = 0.3, \rho_2 = 0.6, v = 0.95, b = 10 \), see Table 4.

### Table 4. Numerical results of \( \phi_j(u) \) for \( b = 10, \theta = 0.5 \).

<table>
<thead>
<tr>
<th>( u )</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi_2(u) )</td>
<td>7.22942 \times 10^{-4}</td>
<td>5.02029 \times 10^{-4}</td>
<td>3.49790 \times 10^{-4}</td>
<td>2.44063 \times 10^{-4}</td>
<td>1.70395 \times 10^{-4}</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( u )</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi_2(u) )</td>
<td>1.8993 \times 10^{-4}</td>
<td>8.31054 \times 10^{-5}</td>
<td>5.80441 \times 10^{-5}</td>
<td>4.05411 \times 10^{-5}</td>
<td>2.83162 \times 10^{-5}</td>
</tr>
</tbody>
</table>
Especially, when \( \alpha(N_1, N_2) = 1, b = 10 \). Fig. 1 and Fig. 2 depict the generating function of the time to ruin \( m_b(u) \) as functions of \( u \). Observing Fig. 1 and Fig. 2, for each fixed \( \theta \) it is easy that a larger \( u \) corresponds to a smaller expected ruin time and \( m_b(u) \) is a decreasing function of \( \theta \) when \( u \) is fixed.

![Fig. 1. Numerical results of \( m_b(u) \) for \( b = 10, u < b \)](image1)

![Fig. 2. Numerical results of \( m_b(u) \) for \( b = 10, u \geq b \)](image2)

5. Conclusion

In this paper, we consider the compound binomial model with general premium rate and a constant dividend barrier. Using the roots of a generalization of Lundberg’s fundamental equation and the general theory on difference equations, we derive an explicit expression for the Gerber-Shiu discounted penalty function up to the time of ruin. In particular, a numerical example is provided to show that the formulae are readily programmable in practice. From the numerical example given above, we can see that the barrier level has a negative effect on the total Gerber-Shiu discounted penalty function.

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Competing Interests

Authors have declared that no competing interests exist.

References


