Modified Maximum Likelihood Estimation for Generalized Exponential Distribution

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Authors’ contributions
This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract
For a Modified Maximum Likelihood Estimate of the parameters of generalized exponential distribution (GE), a hyperbolic approximation is used instead of linear approximation for a function which appears in the Maximum Likelihood equation. This estimate is shown to perform better, in accuracy and simplicity of calculation, than the one based on linear approximation for the same function. Numerical computation for random samples of different sizes from generalized exponential distribution (GE), using type II censoring is done and is shown to be better than that obtained by Lee et al. [1].

Keywords: Asymptotic variance; order statistics; outliers; hyperbolic approximation; censoring; GE.

1 Introduction
Many outlier detection procedures have been given by Balasooriya and Gadag [2], Barnett and Lewis [3], Chikkagoudar and Kunchur [4], Hawkins DM [5], Likes [6], Singh and Lalitha [7], Singh and Lalitha [8] but
When some extreme observations are suspected as outlying, but are not getting detected by any outlier detection test, then a robust procedure to estimate the parameters of the distribution is to use a type II censored sample given by Tiku et al (1986). Here a sample from a generalized exponential distribution is considered, as the ordinary exponential distribution is a particular case of this distribution. The density function of a generalized exponential distribution (GE) is given by

$$g(x; \xi, \sigma) = \xi \sigma (1 - e^{-\sigma x})^{\xi-1} e^{-\sigma x}, \ x, \xi, \sigma > 0,$$

where $\xi$ is the shape parameter and $\sigma$ is the scale parameter.

Gupta and Kundu [9,10] have shown that the use of $GE(\xi, \sigma)$ in analyzing many lifetime data is more effective than any other distribution. Raqab (2002), Raqab and Ahsanullah (2001), Zheng (2003) have worked on some recent developments of a generalized exponential distribution. Another work on this distribution is on the estimation of parameters of a generalized exponential distribution for a complete sample by Gupta and Kundu (2006).

Here the problem of estimation of unknown parameters of a sample from a GE distribution when some of the extreme observations are suspected outliers is considered. For this, a type II censored sample is considered and a modified maximum likelihood estimators are obtained.

Distribution function of a generalized exponential distribution with parameters $\xi$ and $\sigma$ is given by

$$G(x) = (1 - e^{-\sigma x})^\xi, \ \xi, \sigma, x > 0.$$

Let the variable $X$ be transformed to the variable $Z$ as $z = \xi e^{-\sigma x}$.

Then, the Jacobian of transformation is $\xi \sigma e^{-\sigma x}$ and the density function of the transformed variable $Z$ is

$$f(z) = (1 - \frac{z}{\xi})^{\xi-1} \xi, \ z > 0.$$  \hspace{1cm} (2)

The distribution function in terms of the transformed variable is then given by

$$F(z) = -(1 - \frac{z}{\xi})^\xi \xi, \ z > 0.$$  \hspace{1cm} (3)

### 2 Maximum Likelihood Estimation for Censored Sample

Tiku et al (1986) have considered a type II censored data for robust estimation of parameters of several distributions and used a modified maximum likelihood estimators.

In a similar way, in this article, the parameters are estimated using a type II censored sample i.e. a sample obtained after removing $r_1$ and $r_2$ smallest and largest observations respectively.

The likelihood function $L$ of a type II censored sample $Z_{(r_2+1)}, ..., Z_{(n-r_2)}$ is given by

$$L = \frac{n!}{r_1!r_2!} \left( F(z_{r_1+1}) \right)^{r_1} \prod_{i=r_1+1}^{n-r_2} f(z_{(i)}) \left( 1 - F(z_{n-r_2}) \right)^{r_2}.$$  \hspace{1cm} (4)

Rewriting this as

$$L = C_0 \left[ F(z_a) \right]^{r_2} \prod_{i=a}^{b} f(z_{(i)}) \left( 1 - F(z_b) \right)^{r_2},$$

where, $C_0 = \frac{n!}{r_1!r_2!}, \ a = r_1 + 1,$ and $b = n - r_2$.

the log likelihood function is given by
\[ \log L = \log C_0 + r_1 \log F(z_a) + \sum_{i=a}^b \log f(z(i)) + r_2 \log(1 - F(z_b)). \]

For obtaining the likelihood equation for estimating the parameter \( \xi \), \( \log L \) is differentiated with respect to \( \xi \) and equated to zero. Thus,

\[
\frac{\partial \log L}{\partial \xi} = 0,
\]

\[
r_1 \frac{f(z_a)}{F(z_a)} \frac{\partial z_a}{\partial \xi} + \sum_{i=a}^b \frac{1}{f(z(i))} \frac{\partial f(z(i))}{\partial \xi} \frac{\partial z(i)}{\partial \xi} + \frac{r_2}{(1 - F(z_b))} (-f(z_b)) \frac{\partial z_b}{\partial \xi} = 0.
\]

\[
\frac{r_1 f(z_a)}{F(z_a)} \frac{z_a}{\xi} + \sum_{i=a}^b \frac{1}{f(z(i))} \frac{(\xi - 1) \left(1 - \frac{z(i)}{\xi}\right)^{\xi - 2} \left(- \frac{1}{\xi}\right) z(i)}{\xi} - \frac{r_2 f(z_b)}{(1 - F(z_b))} z_b = 0.
\]

\[
\frac{r_2 z_b g_1(z_b)}{\xi} - \frac{(\xi - 1) \sum_{i=a}^b \frac{z(i)}{\xi} (1 - \frac{z(i)}{\xi})}{\xi} r_2 x_b g_2(z_b) = 0.
\]

(3)

where, \( g_1(z) = \frac{f(z)}{F(z)} \) and \( g_2(z) = \frac{f(z)}{1 - F(z)} \).

Similarly, the likelihood equation for estimating the scale parameter \( \sigma \) is given by

\[
\frac{\partial \log L}{\partial \sigma} = 0.
\]

Or

\[
\frac{\partial \log L}{\partial \sigma} = \frac{r_1 f(z_a) z_a}{F(z_a)} \frac{\partial z_a}{\partial \sigma} + \sum_{i=a}^b \frac{1}{f(z(i))} \frac{\partial f(z(i))}{\partial \sigma} \frac{\partial z(i)}{\partial \sigma} + \frac{r_2}{(1 - F(z_b))} (-f(z_b)) \frac{\partial z_b}{\partial \sigma}
\]

\[
= r_1 g_1(z_a) z_a + \sum_{i=a}^b \frac{1}{f(z(i))} (\xi - 1) \left(1 - \frac{z(i)}{\xi}\right)^{\xi - 2} \left(- \frac{1}{\xi}\right) z(i) - r_2 g_2(z_b) x_b z_b = 0.
\]

Or

\[
= -r_1 g_1(z_a) x_a z_a + \frac{(\xi - 1) \sum_{i=a}^b z(i) x(i)}{(1 - \xi)} + r_2 g_2(z_b) x_b z_b = 0.
\]

(4)

Solving (3) and (4) for \( \xi \) and \( \sigma \) explicitly are very cumbersome if not impossible. Hence modifications of these equations are sought.

### 3 Modified Maximum Likelihood Estimation for Left Censored Sample

When some of the suspected contaminant observations are the smaller observations of the sample then they are deleted from the sample and a left censored sample is obtained. Let \( r_1 \) be the number of observations that are censored from left side.

Let \( z_{(r_1+1)} \leq \cdots \leq z_{(n)} \) be \( (n - r_1) \) order statistics, then the likelihood equations are given by-

\[
\frac{r_1 x(a) g_1(x(a))}{\xi} + \frac{(\xi - 1)}{\xi^2} \sum_{i=a}^n z(i) g_1(z(i)) = 0
\]

(5)

and

\[
- r_1 g_1(\xi e^{-\sigma x(a)}) (\xi e^{-\sigma x(a)} x(a)) + \sum_{i=a}^n \frac{(\xi - 1) \xi e^{-\sigma x(i)} x(i)}{(1 - e^{-\sigma x(a)})} = 0.
\]

(6)

The equations (5) and (6) have no explicit solution and solving of these equations are very difficult due to function \( g_1(z) \). To solve these equations, some iterative methods have been developed by Cohen (1957, 1961) and Harter and Moore (1966). Due to implicit nature of these iterations, it is not very useful.

Lee et al (1980) and Tiku (1986) have given MMLE by using a linear approximation
\[ \alpha + \beta z \] to the function \( g_1(z) \). But since this approximation is not so good for exponential distributions, Lalitha and Mishra [11] have given MMLE using the hyperbolic approximation \( z, g(z) \cong k \) and have shown this to be a better approximation.

Another approximation to the function \( g_1(z) \) was given by Kumar (2009) for the case of three parameter gamma distribution, which is

\[ z, (g_1(z) + a_1) \cong k_1, \]  

where \( k_1 \) and \( a_1 \) can be obtained by considering any two points \( h_1 \) and \( h_2 \) on the curve which are very close to each other so that they tend to a common value \( h \) as \( n \to \infty \). This approximation, when applied to a generalized exponential distribution, is shown to be the best out of all approximations in the following discussion.

To check the closeness of the approximation, the values of \( k_1 \) and \( a_1 \) have to be determined, which is done by considering any two points \( h_1 \) and \( h_2 \) on the curve which are very close to each other so that they tend to a common value \( h \) as \( n \to \infty \).

Then

\[ \frac{g(h_1)-g(h_2)}{h_2-h_1} = \frac{k_1}{h_1 h_2} \frac{k_1}{h^2}, \]

where \( h \) is given by \( q_1 \), and \( q_1 = r_1/n \) as per David and Nagaraja (2003).

Hence, the above equation reduces to

\[ k_1 = -h^2 [(d/dz) g_1(z)]_{z=h}. \]  

From (8), the value of \( a_1 \) was

\[ a_1 = \frac{1}{h} [k_1 - h g_1(z)], \]

which can be solved by substituting the value of \( k_1 \) given in (8).

The approximated values and the actual values of \( g_1(z) \) are tabulated in Table 1 for the location parameter \( \xi = 0.75, 1.5, 2.5 \) and10. These are also depicted graphically in Figs. 1 and 2 for \( \xi = 1.5 \) and 2.5 respectively.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
\multicolumn{2}{|c|}{\( z \)} & \multicolumn{2}{|c|}{Hyperbolic Approximation of \( g_1(z) \)} & \multicolumn{2}{|c|}{Actual value of \( g_1(z) \)} \\
\hline
\multicolumn{2}{|c|}{\( \xi = 0.75, \sigma = 1 \)} & \multicolumn{2}{|c|}{\( \xi = 1.5, \sigma = 1.5 \)} & \multicolumn{2}{|c|}{\( \xi = 2.5, \sigma = 2 \)} & \multicolumn{2}{|c|}{\( \xi = 10, \sigma = 2.5 \)} \\
\hline
\multicolumn{2}{|c|}{\( z \)} & \multicolumn{2}{|c|}{\( \xi = 0.75, \sigma = 1 \)} & \multicolumn{2}{|c|}{\( \xi = 1.5, \sigma = 1.5 \)} & \multicolumn{2}{|c|}{\( \xi = 2.5, \sigma = 2 \)} & \multicolumn{2}{|c|}{\( \xi = 10, \sigma = 2.5 \)} \\
\hline
2 & 0.5348 & 0.5324 & 3 & 1.9762 & 1.9525 \\
2.5 & 0.8984 & 0.8913 & 3.5 & 1.4527 & 1.4031 \\
3 & 1.0432 & 1.0401 & 4 & 1.09 & 1.11 \\
3.5 & 1.2562 & 1.2511 & 4.5 & 0.491 & 0.517 \\
\hline
5.5 & 10.0957 & 10.0925 & 7 & 10.7932 & 10.7921 \\
6 & 5.1357 & 5.1318 & 7.5 & 8.8492 & 8.6322 \\
6.5 & 3.2543 & 3.2524 & 8 & 5.3258 & 5.1349 \\
7 & 2.5683 & 2.5642 & 8.5 & 3.1183 & 3.2398 \\
\hline
\end{tabular}
\caption{Hyperbolic approximation of \( g_1(z) \) and actual value of \( g_1(z) \)}
\end{table}
Fig. 1. Plot of actual and the approximate values of $g_1(z)$ with $\xi = 1.5$

Fig. 2. Plot of actual and the approximate values of $g_1(z)$ with $\xi = 2.5$

It can be seen from above Table 1 as well as Fig. 1, 2 that approximation given in equation (7) is good enough.
4 Estimation of Parameters Using the Approximations for Left Censored Sample

For estimating the parameters of a Generalized exponential distribution using a left censored sample with \( r_1 \) observations censored on the left side, equations (5) and (6) were solved for \( \xi \) and \( \sigma \) by replacing the function \( g_1(x) \) by the approximate function given by (7).

Thus, using the approximation (7) of the function \( g_1(x) \), equation (5) becomes

\[
\frac{r_1(x) g_1(z(x))}{\xi} + \frac{(\xi - 1)}{\xi} \sum_{i=0}^{n} z(i) g_1(z(i)) = 0.
\]

\[
r_1(z(x)) g_1(z(x)) + \frac{(\xi - 1)}{\xi} \sum_{i=0}^{n} z(i) g_1(z(i)) = 0.
\]

\[
r_1(k_1 - a_1 z(x)) + \frac{(\xi - 1)}{\xi} \sum_{i=0}^{n} (k_1 - a_1 z(i)) = 0.
\]

\[
r_1(k_1 - a_1 z(x)) + \frac{(\xi - 1)}{\xi} \{ (n - a + 1)k_1 - \sum_{i=0}^{n} a_1 z(i) \} = 0.
\]

\[
\xi r_1(k_1 - a_1 \xi e^{-\sigma x(i)}) + (\xi - 1) \{ (n - r_1) - \sum_{i=0}^{n} a_1 \xi e^{-\sigma x(i)} \} = 0.
\]

\[
\xi^2 \{ a_1 r_1 e^{-\sigma x(i)} + \sum_{i=0}^{n} a_1 e^{-\sigma x(i)} \} - \xi \{ n K_1 + \sum_{i=0}^{n} a_1 e^{-\sigma x(i)} \} + (n - r_1) K_1 = 0
\]

\[
\Rightarrow \quad \xi = \frac{B + \sqrt{B^2 - 4AC}}{2A}, \quad (9)
\]

where,

\[A = a_1 r_1 e^{-\sigma x(i)} + \sum_{i=0}^{n} a_1 e^{-\sigma x(i)}, \quad B = n K_1 + \sum_{i=0}^{n} a_1 e^{-\sigma x(i)} \]

\[C = (n - r_1) K_1.\]

Now for estimating \( \sigma \), the modified maximum likelihood equation (6) becomes

\[
-r_1 g_1(x) (z(x) x(i)) + \sum_{i=0}^{n} \frac{(\xi - 1)}{\xi} \frac{z(i) x(i)}{1 - \alpha x(i)} = 0.
\]

\[
r_1 x(i) (k_1 - a_1 z(x)) - \sum_{i=0}^{n} \frac{(\xi - 1)}{\xi} z(i) x(i) = 0.
\]

\[
\xi r_1 x(i) (k_1 - a_1 \xi (1 - \sigma x(i))) + (\xi - 1) \sum_{i=0}^{n} (k_1 - a_1 \xi (1 - \sigma x(i))) x(i) = 0.
\]

\[
\bar{\sigma} = \frac{P}{Q}.
\]

(10)

where, \( P = \xi r_1 x(i) (k_1 - a_1 \xi) + (\xi - 1) \sum_{i=0}^{n} (k_1 x(i) - a_1 \xi x(i)) \) and \( Q = -\xi^2 r_1 x^2(i) a_1 - (\xi - 1) \sum_{i=0}^{n} a_1 \xi x^2(i). \)

The above equation was solved for \( \sigma \) and an estimate \( \bar{\sigma} \) of \( \sigma \) was obtained using the following steps.

- **Step 1:** Firstly, \( \bar{\xi} \) was calculated using (9).

- **Step 2:** Using the value of \( \bar{\xi} \) obtained in step 1, \( \bar{\sigma} \) was obtained using equation (10) using Newton-Raphson method with an accuracy limit of \( 10^{-4} \).

- **Step 3:** Steps 1-2 were repeated 10,000 times.

- **Step 4:** MML estimates of \( \xi \) and \( \sigma \) are then the means of the values of \( \xi \) and \( \sigma \) obtained in step 3. The estimates from left censored samples along with their mean square error (MSE) are shown in Table 2 for \( r_1 = 1 \) and in Table 3 for \( r_1 = 2 \).
Table 2. Estimates of parameters from a left censored sample along with their mean square error (MSE) for $r_1=1$

<table>
<thead>
<tr>
<th>Sample size n</th>
<th>$\xi=0.5, \sigma=1$</th>
<th>$\xi=2, \sigma=1.5$</th>
<th>$\xi=5, \sigma=2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\xi}$</td>
<td>$\hat{\sigma}$</td>
<td>$\hat{\xi}$</td>
<td>$\hat{\sigma}$</td>
</tr>
<tr>
<td>10</td>
<td>0.303255</td>
<td>0.47776</td>
<td>0.8913</td>
</tr>
<tr>
<td>$MSE$</td>
<td>0.00387086</td>
<td>0.027273</td>
<td>0.122921</td>
</tr>
<tr>
<td>20</td>
<td>0.3241</td>
<td>0.65942</td>
<td>0.9282</td>
</tr>
<tr>
<td>$MSE$</td>
<td>0.00154704</td>
<td>0.0058</td>
<td>0.057438</td>
</tr>
<tr>
<td>50</td>
<td>0.41479</td>
<td>0.8355</td>
<td>0.95166</td>
</tr>
<tr>
<td>$MSE$</td>
<td>0.00014521</td>
<td>0.000541</td>
<td>0.02198</td>
</tr>
<tr>
<td>100</td>
<td>0.48817</td>
<td>0.85395</td>
<td>0.96431</td>
</tr>
<tr>
<td>$MSE$</td>
<td>0.000001</td>
<td>0.000213</td>
<td>0.010726</td>
</tr>
</tbody>
</table>

Table 3. Estimates of parameters from a left censored sample along with their mean square error (MSE) for $r_1=2$

<table>
<thead>
<tr>
<th>Sample size n</th>
<th>$\xi=0.5, \sigma=1$</th>
<th>$\xi=2, \sigma=1.5$</th>
<th>$\xi=5, \sigma=2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\xi}$</td>
<td>$\hat{\sigma}$</td>
<td>$\hat{\xi}$</td>
<td>$\hat{\sigma}$</td>
</tr>
<tr>
<td>10</td>
<td>0.30062</td>
<td>0.229119</td>
<td>0.90307</td>
</tr>
<tr>
<td>$MSE$</td>
<td>0.003975</td>
<td>0.05942573</td>
<td>0.12032554</td>
</tr>
<tr>
<td>20</td>
<td>0.39264</td>
<td>0.434372</td>
<td>0.930912</td>
</tr>
<tr>
<td>$MSE$</td>
<td>0.000576</td>
<td>0.01599675</td>
<td>0.05714746</td>
</tr>
<tr>
<td>50</td>
<td>0.46779</td>
<td>0.555473</td>
<td>0.955828</td>
</tr>
<tr>
<td>$MSE$</td>
<td>0.0000207</td>
<td>0.00395209</td>
<td>0.0218059</td>
</tr>
<tr>
<td>100</td>
<td>0.4991</td>
<td>0.823561</td>
<td>1.02789</td>
</tr>
<tr>
<td>$MSE$</td>
<td>0.000000008</td>
<td>0.00031313</td>
<td>0.00944998</td>
</tr>
</tbody>
</table>

It is observed that MSE is large for small sample sizes in these tables. The MSE of the estimates of the parameters decreases as sample sizes increases. It is also clear from the tables that the larger the value of $r_1$, the MSE of the estimates are larger.

5 Modified Maximum Likelihood Estimation for Right Censored Sample

When some of the suspected contaminant observations are the larger observations of the sample, these are deleted and a right censored sample is obtained. Let $r_2$ be the number of observations that are censored on the right side.

Let $z_{(1)} \leq \cdots \leq z_{(n-r_2)}$ be $(n-r_2)$ order statistics of the censored sample of size $(n-r_2)$. Then the likelihood equations in this case are given by

\[
\frac{r_2z_{(b)}g_2(z_{(b)})}{\xi} + \frac{(\xi-1)}{\xi^2} \sum_{i=1}^{b} \frac{z_{(i)}}{1 - \frac{2z_{(i)}}{\xi}} = 0 \tag{11}
\]

and

\[
\sum_{i=1}^{b} \frac{(\xi-1)}{\xi} \frac{e^{-\sigma X_{(i)}}}{1 - e^{-\sigma X_{(i)}}} + r_2 g_2(e^{-\sigma X_{(b)}})(e^{-\sigma X_{(b)}}X_{(b)}) = 0. \tag{12}
\]

Due to involvement of the function $g_2(z) = \frac{f(z)}{1-F(z)}$ in the likelihood equations for $\xi$ and $\sigma$, finding a solution of the above system of the equations is rather difficult. Hence, $g_2(z)$ was approximated by a simpler function of the form

\[
z \cdot (g_2(z) - a_2) = k_2. \tag{13}
\]
where $k_2$ and $a_2$ can be obtained by considering any two points $h_1$ and $h_2$ on the curve which are very close to each other so that they tend to common value $h$ as $n \to \infty$.

Then

$$\frac{g(h_1) - g(h_2)}{h_2 - h_1} = \frac{k_2}{h_1 h_2},$$

where $h$ is given by

$$h = 1 - q_2, \quad (q_2 = r_2/n) \quad [\text{David and Nagaraja (2003)}]$$

and the above equation reduces to

$$k_2 = -h^2[(d/dz). g_2(z)]_{z=h}, \quad a_2 = \frac{1}{h} [h g_2(z) - k_2].$$

It can be easily verified that the function $g_2(z)$ over an interval of finite length lies very close to the curve.

The approximated values and the actual values of $g_2(z)$ are tabulated in Table 3 for the location parameter $\xi = 1, 3, 5$ and 7. These are also depicted graphically in Figs. 3 and 4 for $\xi = 5$ and 7 respectively.

![Fig. 3. Plot of actual and the approximate values of $g_2(z)$ with $\xi = 5$](image1)

![Fig. 4. Plot of actual and the approximate values of $g_2(z)$ with $\xi = 7$](image2)

It can be seen from Table (4) as well as fig. 3, 4 that approximation given in equation (13) is good enough. For better approximation other approximations were tried for $g_2(z)$ i.e. $g_2(z) \equiv k_2$. It can be seen from the
following fig. 5, 6 that approximation $g_2(z) \approx k_2$ is also good but it is simpler than other. The approximated values and the actual values of $g_2(z)$ are tabulated in Table 5 for the location parameter $\xi = 1, 3, 5$ and 7.

Table 4. Hyperbolic Approximation of $g_2(z)$ and actual values of $g_2(z)$

<table>
<thead>
<tr>
<th>$z$</th>
<th>Hyperbolic Approximation of $g_2(z)$</th>
<th>Actual value of $g_2(z)$</th>
<th>$z$</th>
<th>Hyperbolic Approximation of $g_2(z)$</th>
<th>Actual value of $g_2(z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.3263158</td>
<td>0.5263158</td>
<td>0.1</td>
<td>0.5648166</td>
<td>0.4909611</td>
</tr>
<tr>
<td>0.2</td>
<td>0.5555556</td>
<td>0.5555556</td>
<td>0.2</td>
<td>0.4804707</td>
<td>0.4804707</td>
</tr>
<tr>
<td>0.5</td>
<td>0.7866667</td>
<td>0.6666667</td>
<td>0.5</td>
<td>0.3955694</td>
<td>0.4398827</td>
</tr>
<tr>
<td>1</td>
<td>1.16</td>
<td>1</td>
<td>1</td>
<td>0.2837728</td>
<td>0.3428571</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$z$</th>
<th>Hyperbolic Approximation of $g_2(z)$</th>
<th>Actual value of $g_2(z)$</th>
<th>$z$</th>
<th>Hyperbolic Approximation of $g_2(z)$</th>
<th>Actual value of $g_2(z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.597159</td>
<td>0.4844572</td>
<td>0.1</td>
<td>0.6098442</td>
<td>0.4817227</td>
</tr>
<tr>
<td>0.2</td>
<td>0.467864</td>
<td>0.4678635</td>
<td>0.2</td>
<td>0.4626641</td>
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<tr>
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</tr>
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</table>

Fig. 5. Plot of actual and the approximate values of $g_2(z)$ with $\xi = 5$
6 Estimation of Parameters Using the Approximations for Right Censored Sample

If the number of observations $r_2$, are censored on right side, then the likelihood equations become as

$$\frac{r_2 g_2(x(b))}{\xi} + \frac{(\xi - 1)}{\xi^2} \sum_{i=1}^{b} \frac{x(i)}{1 - \xi x(i)} = 0 \quad (14)$$

and

$$\sum_{i=1}^{b} \frac{(\xi - 1)}{\xi} \frac{x(i)x(i)}{1 - \xi x(i)} + r_2 g_2(x(b))x(b) = 0. \quad (15)$$

On using the approximation $g_2(x) \approx k_2$ of the function $g_2(x)$, in equation (14) and equating to zero, it reduces to

$$\frac{r_2 g_2(x(b))}{\xi} = -\frac{(\xi - 1)}{\xi^2} \sum_{i=1}^{b} \frac{x(i)}{1 - \xi x(i)}$$

$$r_2 k_2 e^{-\sigma x(b)} = -\frac{(\xi - 1)}{\xi} \sum_{i=1}^{b} \frac{e^{-\sigma x(i)}}{1 - e^{-\sigma x(i)}}.$$ 

$$\xi = \frac{1}{1 + (r_2 k_2 e^{-\sigma x(b)} / \sum_{i=1}^{b} \frac{e^{-\sigma x(i)}/(1 - e^{-\sigma x(i)})})}. \quad (16)$$

Again using the approximation $g_2(x) \approx k_2$ of the function $g_2(x)$, in equation (15) and equating to zero,

$$-r_2 k_2 x(b) x(b) = \frac{(\xi - 1)}{\xi} \sum_{i=1}^{b} \frac{x(i)x(i)}{1 - \xi x(i)}.$$ 

$$-r_2 k_2 x(b) x(b) = \frac{(\xi - 1)}{\xi} \sum_{i=1}^{b} \frac{\xi e^{-\sigma x(i)} x(i)}{1 - e^{-\sigma x(i)}}.$$ 

$$-r_2 k_2 \xi e^{-\sigma x(b)} x(b) = (\xi - 1) \sum_{i=1}^{b} \frac{x(i)}{e^{\sigma x(i) - 1}}.$$
\(-r_2 k_x (1 - \sigma x_b) = \frac{(\xi - 1) \beta}{\sigma}\).

\((x_b)^2 r_2 k_x \sigma^2 - r_2 k_x (x_b) \sigma - (\xi - 1) b = 0\).

This gives \(\hat{\sigma} = \frac{b_1 + \sqrt{b_1^2 - 4 b_2 c_1}}{2 b_1}\). (17)

where, \(A_1 = (x_b)^2 r_2 k_x \beta, B_1 = r_2 k_x (x_b) \sigma\) and \(C_1 = -(\xi - 1) b\).

The above equation was solved for \(\sigma\) and an estimate \(\hat{\sigma}\) of \(\sigma\) was obtained using the following steps.

- **Step 1:** Firstly, \(\xi\) was calculated using (16).
- **Step 2:** Using the value of \(\xi\) obtained in step 1, \(\hat{\sigma}\) was obtained using equation (17) using Newton-Raphson method with an accuracy limit of 10^-4.
- **Step 3:** Steps 1-2 were repeated 10,000 times.
- **Step 4:** MML estimates of \(\xi\) and \(\sigma\) are then the means of the values of \(\xi\) and \(\sigma\) obtained in step 3. The estimates from right censored samples along with their mean square error (MSE) are shown in Table 6 for \(r_2 = 1\) and in Table 7 for \(r_2 = 2\).

**Table 6. Estimates of parameters from a right censored sample along with their mean square error (MSE) for \(r_2 = 1\)**

<table>
<thead>
<tr>
<th>Sample size n</th>
<th>(\xi = 0.5, \sigma = 1)</th>
<th>(\xi = 2, \sigma = 1.5)</th>
<th>(\xi = 5, \sigma = 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\hat{\xi})</td>
<td>(\hat{\sigma})</td>
<td>(\hat{\xi})</td>
<td>(\hat{\sigma})</td>
</tr>
<tr>
<td>10</td>
<td>0.28273</td>
<td>0.6527</td>
<td>1.20999</td>
</tr>
<tr>
<td><strong>MSE</strong></td>
<td>0.00472063</td>
<td>0.012062</td>
<td>0.062411</td>
</tr>
<tr>
<td>20</td>
<td>0.361332</td>
<td>0.7867</td>
<td>1.3149</td>
</tr>
<tr>
<td><strong>MSE</strong></td>
<td>0.00096144</td>
<td>0.002275</td>
<td>0.023468</td>
</tr>
<tr>
<td>50</td>
<td>0.471064</td>
<td>0.899</td>
<td>1.49016</td>
</tr>
<tr>
<td><strong>MSE</strong></td>
<td>0.000017</td>
<td>0.000204</td>
<td>0.005199</td>
</tr>
<tr>
<td>100</td>
<td>0.495</td>
<td>0.94595</td>
<td>1.86821</td>
</tr>
<tr>
<td><strong>MSE</strong></td>
<td>0.00000025</td>
<td>0.000029</td>
<td>0.000174</td>
</tr>
</tbody>
</table>

**Table 7. Estimates of parameters from a right censored sample along with their mean square error (MSE) for \(r_2 = 2\)**

<table>
<thead>
<tr>
<th>Sample size n</th>
<th>(\xi = 0.5, \sigma = 1)</th>
<th>(\xi = 2, \sigma = 1.5)</th>
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<tbody>
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<td>(\hat{\sigma})</td>
<td>(\hat{\xi})</td>
<td>(\hat{\sigma})</td>
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<tr>
<td>10</td>
<td>0.287451</td>
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<td>0.18214</td>
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<tr>
<td><strong>MSE</strong></td>
<td>0.0045177</td>
<td>0.065416</td>
<td>0.330462</td>
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<tr>
<td>20</td>
<td>0.388811</td>
<td>0.29993</td>
<td>1.26443</td>
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<tr>
<td><strong>MSE</strong></td>
<td>0.00006181</td>
<td>0.024505</td>
<td>0.027053</td>
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<tr>
<td>50</td>
<td>0.419227</td>
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<td>1.36572</td>
</tr>
<tr>
<td><strong>MSE</strong></td>
<td>0.000130</td>
<td>0.004810</td>
<td>0.008046</td>
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<tr>
<td>100</td>
<td>0.489457</td>
<td>0.6103</td>
<td>1.61004</td>
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<tr>
<td><strong>MSE</strong></td>
<td>0.0000011</td>
<td>0.001519</td>
<td>0.001521</td>
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Larger MSE can be seen from the estimates of each parameter for small samples sizes. However MSE decreases as sample size increases. It can also be seen that for \(r_2 = 2\), the MSE of estimates of each parameter is higher for \(n=10, 20\) than the MSE of estimates for the case of \(r_2 = 1\).

**7 Conclusion**

This procedure converges more rapidly for small samples than the linear approximation used by Lee et al. [1] thus this hyperbolic approximation can be recommended on the whole for Generalized exponential distribution.
Competing Interests

Authors have declared that no competing interests exist.

References


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