The Inverse Lomax-G Family with application to Breaking Strength Data

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Authors contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

We proposed a new class of distributions with two additional positive parameters called the Inverse Lomax-G (IL-G) class. A special case was discussed, by taking Weibull as a baseline. Different properties of the new family that hold for any type of baseline model are derived including moments, moment generating function, entropy for Renyi, entropy for Shannon, and order statistics. The performances of the maximum likelihood estimates of the parameters of the sub-model of the Inverse Lomax-G family were evaluated through a simulation study. Application of the sub-model to the Breaking strength data clearly showed its superiority over the other competing models.

Keywords: Entropy; new Weibull inverse Lomax; complete sample; Monte Carlo simulation; inverse lomax; inverse Lomax-G family.

Mathematical Subject Classification: 60E05, 62E15.

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1 Introduction

The distribution of Inverse Lomax (IL) is part of a distribution of Beta forms. Other family members include, among others, Pareto (1), logistics, Dagum, generalized second-type beta distributions, and Singh maddala \cite{1}. In several fields, such as Actuarial Science and Economics, IL distribution has since gained more attention (see \cite{1}), Geophysical data (see \cite{2}), Survival analysis (see \cite{3}), and Medical Science (see \cite{4} and \cite{5}).

Some attempts have been made to define new families of probability distributions which strengthen well-known distribution families while providing greater flexibility in practical data modeling. Following from the T-X approach by \cite{6}, we define the cumulative distribution function (cdf) of the new family of distributions as

\[ F(x) = \int_b^{\mathcal{N}(H(x))} l(t_1)dt_1 \]  

(1.1)

where \( N(H(x)) \) is the function of the baseline cdf \( H(x) \) of any random variable (RV) \( X \) that satisfies the following conditions below:

(a) \( N(H(x)) \in [b, c] \)

(b) \( N(H(x)) \) is non-decreasing, monotonically differentiable

(c) \( N(H(x)) \Rightarrow b \) as \( x \Rightarrow -\infty \) and \( N(H(x)) \Rightarrow c \) as \( x \Rightarrow \infty \). Let \( T_1 \) be a RV which is continuous with a probability density function (pdf) \( l(t_1) \) defined on the closed interval \([b, c]\).

Some of the generalized families of distributions based on this approach in the literature include: Weibull G family by \cite{7}, Lomax Generator of distributions by \cite{8}, Odd Generalized Exponential family by \cite{9}, Odd Lindley-G family by \cite{10}, Gompertz-G family by \cite{11}, Odd Frechet G family by \cite{12}, Power Lindley G family by \cite{13}, Topp Leone Exponentiated-G by \cite{14}, and Odd Chen-G family by \cite{15}.

Inverse Lomax (IL) distribution has both scale and shape parameters which makes it more flexible in modeling datasets. In order to increase its flexibility and usage, we wish to generalize the IL model. The probability density function (pdf) and cdf of the IL distribution are given by

\[ h(x; \alpha, \beta) = \frac{\alpha \beta}{x^2} \left( 1 + \frac{\beta}{x} \right)^{-(\alpha - 1)} \]  

(1.2)

\[ H(x; \alpha, \beta) = \left( 1 + \frac{\beta}{x} \right)^{-(\alpha)} \quad x > 0, \alpha, \beta > 0 \]  

(1.3)

The objective of this paper is to propose a new family of distributions called the Inverse Lomax-G family of distributions which has the capacity of providing more robust compound probability distributions when modeling real life data set. This new family adds two additional parameters to the baseline distribution. This article is structured as follows: The Inverse Lomax-G family (IL-G) is defined in Sec. 2. In Sec. 3 a special sub-model of the IL-G is derived. A mixture representation of the cdf of the IL-G is presented in Sec. 4, while some of the IL-G’s properties including the quantile function, order statistics, moments, moment generating function, and Entropies are discussed in Sec. 5. In Sec. 6, the estimation of the parameters of the IL-G is conducted using the Maximum Likelihood method. A Monte Carlo simulation study is used to estimate the bias and mean squared error of the MLE estimates of the parameters of the sub-model in Sec. 7. The application of the Inverse Lomax Weibull (ILW) and its competitors to the Breaking strength data is discussed in Sec. 8, while Sec. 9 concludes the paper.
2 The Inverse Lomax-G Family

In this section, we derive the distribution of the Inverse Lomax-G Family. The pdf, cdf, hazard function (hf), reversed hazard function, survival function, and cumulative hf are displayed.

Let $H(x; v)$ and $h(x; v)$ be the baseline cdf and pdf, and $v$ be a vector of parameters, let $l(t_1)$ be as defined in Eqn. 1.2, then the cdf $F(x; \alpha, \theta) = F(x; \vartheta)$ of the IL-G family of distribution is defined as

$$F(x; \vartheta) = \int_0^{\frac{l(t_1)}{H(x; v)}} l(t_1)dt_1 = \left(1 + \frac{\beta H(x; v)}{H(x; v)}\right)^{-\alpha}; \quad x > 0, \alpha, \beta, v > 0 \quad (2.1)$$

where $\vartheta = (\alpha, \beta, v)^T$, $H(x; v) = 1 - H(x; v)$ and also $\alpha$ and $\beta$ are the two additional parameters that are added to make the baseline distribution much more flexible. The corresponding pdf $f(x; \vartheta)$ of IL-G family obtained by differentiating Eqn. 2.1 as given below

$$f(x; \vartheta) = \frac{\beta ah(x; v)}{H(x; v)^2} \left(1 + \frac{\beta H(x; v)}{H(x; v)}\right)^{-(1+\alpha)} \quad (2.2)$$

The hazard function (hf), reversed hazard function rhsf(x), survival function (s), and cumulative hazard functions $C(x)$ are given below:

$$h_f(x; \vartheta) = \frac{\alpha \beta h(x; v)}{H(x; v)^2} \left(1 + \frac{\beta H(x; v)}{H(x; v)}\right)^{-(1+\alpha)} \left[1 - \left(1 + \frac{\beta H(x; v)}{H(x; v)}\right)^{-\alpha}\right]^{-1} \quad (2.3)$$

$$r(x; \vartheta) = \frac{\alpha \beta h(x; v)}{H(x; v)^2} \left(1 + \frac{\beta H(x; v)}{H(x; v)}\right)^{-(1+\alpha)} \quad (2.4)$$

$$s(x; \vartheta) = 1 - \left(1 + \frac{\beta H(x; v)}{H(x; v)}\right)^{-\alpha} \quad (2.5)$$

$$C(x; \vartheta) = \int_{-\infty}^{x} p(q) dq = -\log[s(x)] = -\log \left[1 - \left(1 + \frac{\beta H(x; v)}{H(x; v)}\right)^{-\alpha}\right] \quad (2.6)$$

The quantile function (qf) of IL-G family can be derived by inverting Eqn. 2.1 as given below

$$Q(U) = H^{-1} \left\{ \beta \left[U^{(1+\alpha)} - 1\right] \right\} \quad (2.7)$$

where $H^{-1}(.)$ is quantile function (qf) of any baseline distribution, $U$ is uniformly distributed i.e. $U \sim U(0, 1)$, and Eqn. 2.7 can be used to draw samples for the purpose of simulation studies.

3 Special Sub model

Suppose that the parent distribution is Weibull. Then $h(x; \theta, \lambda) = \lambda \theta x^{\theta-1} \exp\{-\lambda x^{\theta}\}$ and $H(x; \theta, \lambda) = 1 - \exp\{-\lambda x^{\theta}\}$. With $x > 0$, $\lambda, \theta > 0$, the Inverse Lomax Weibull (ILW) distribution has the cdf given as

$$F_{ILW}(x) = \left(1 + \frac{\beta \exp\{\lambda x^{\theta}\}}{\exp\{\lambda x^{\theta}\} - 1}\right)^{-\alpha} \quad (3.1)$$

The ILW distribution becomes Inverse Lomax Exponential (ILExp) distribution when $\theta = 1$ and an Inverse Lomax Rayleigh (ILR) distribution when $\theta = 2$. The corresponding pdf of Eqn. (3.1) is given below:

$$f_{ILW}(x) = \frac{\alpha \beta \lambda x^{\theta-1} \exp\{-\lambda x^{\theta}\}}{[1 - \exp\{-\lambda x^{\theta}\}]^2} \left(1 + \frac{\beta \exp\{\lambda x^{\theta}\}}{\exp\{\lambda x^{\theta}\} - 1}\right)^{-(1+\alpha)} \quad (3.2)$$
The quantile function, rhf(x), C(x), and hf(x), are given by

\[ Q(U) = \left[ \log \left( \frac{u^{-1}(\Phi + \beta)}{u^{-1}(\Phi + \lambda)} \right) \right]^{\frac{1}{\beta}} \]  

(3.3)

\[ h_{ILW}(x) = \frac{\theta \lambda \alpha \beta x^{\theta-1} \exp\{-\lambda x^\theta\} \left(1 + \frac{\beta}{\exp(\lambda x^\theta) - 1}\right)^{-(1+\alpha)}}{1 - \left(1 + \frac{\beta}{\exp(\lambda x^\theta) - 1}\right)^{-\alpha} \left[1 - \exp\{-\lambda x^\theta\}\right]^2} \]  

(3.4)

\[ H_{ILW}(x) = -\log \left[ \left(1 + \frac{\beta}{\exp(\lambda x^\theta) - 1}\right)^{-\alpha} \right] \]  

(3.5)

\[ r_{ILW}(x) = \frac{\theta \alpha \lambda \beta x^{\theta-1} \exp\{-\lambda x^\theta\} \left(1 + \frac{\beta}{\exp(\lambda x^\theta) - 1}\right)^{-(1+\alpha)}}{\left(1 + \frac{\beta}{\exp(\lambda x^\theta) - 1}\right)^{-\alpha} \left[1 - \exp\{-\lambda x^\theta\}\right]^2} \]  

(3.6)

Fig. 1. illustrates different density and hazard forms of the ILW distribution. By fixing the parameters of the baseline distribution and varying the additional two parameters, \(\alpha\) and \(\beta\), the plots show the additional flexibility induced by the addition of these two parameters to the weibull distribution.

4 Mixture Representations

Here, we present the power series expansion of the IL-G family by expanding Eqt. 2.1. using the binomial expansion

\[ (1 + x)^{-t} = \sum_{c=0}^{\infty} \binom{-t}{c} x^{-t-c} \]
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\[ F(x; \theta) = \left(1 + \frac{\beta \bar{H}(x; v)}{H(x; v)}\right)^{-\alpha} = \sum_{i=0}^{\infty} \left(\frac{-\alpha}{i}\right) \left(\frac{\beta \bar{H}(x; v)}{H(x; v)}\right)^{(\alpha+i)} \]  

(4.1)

\[ F(x; \theta) = \sum_{i=0}^{\infty} \left(-\alpha\right) \beta^{-(\alpha+i)} \bar{H}(x; v)^{-(\alpha+i)} H(x; v)^{(\alpha+i)} \]  

since

\[ \bar{H}(x; v)^{-(\alpha+i)} = \sum_{j=0}^{\infty} \frac{\Gamma(\alpha+j+i)}{j! \Gamma(\alpha+i)} H(x; v)^{j} \]

Then, Eqt. (4.2) can also be written as

\[ F(x; \theta) = \sum_{i,j=0}^{\infty} \left(-\alpha\right) \frac{\Gamma(\alpha+j+i)}{j! \Gamma(\alpha+i)} \beta^{-(\alpha+i)} \left(\frac{\gamma}{1} + \frac{i}{1}\right)^{\gamma} \left(\frac{1}{1} + \frac{j}{1}\right) \]  

(4.3)

where \( \sigma_{(i,j)}(x; v) = \left(-\alpha\right) \frac{\Gamma(\alpha+j+i)}{j! \Gamma(\alpha+i)} \beta^{-(\alpha+i)} \) and \( \Pi_{(i,j)}(x; v) \) is the exp-G family cdf with parameter power \( (i + \alpha + j) \). The corresponding IL-G pdf is given by

\[ f(x; \theta) = \sum_{i,j=0}^{\infty} \sigma_{(i,j)} \pi_{(i,j)}(x; v) \]  

(4.4)

where \( \pi_{(i,j)}(x; v) = (\alpha + i + j)h(x; v)H(x; v)^{(\alpha+i+j-1)} \).

5 Mathematical Properties of IL-G

We derived some of the mathematical properties of the IL-G family.

5.1 Moments and Moments Generating Function (mgf)

Suppose the random variable \( X \) comes from IL-G family with parameter space \( \theta \), then the \( r^{th} \) moment is given by

\[ E(X^r) = \int_{0}^{\infty} x^r f(x) dx = \int_{0}^{\infty} x^r \sum_{i,j=0}^{\infty} \sigma_{(i,j)} \pi_{(i,j)}(x; v) dx \]  

(5.1)

\[ = \sum_{i,j=0}^{\infty} \sigma_{(i,j)} \int_{0}^{\infty} x^r \left(\alpha + j + i\right)h(x; v)H(x; v)^{(i+j+\alpha-1)} dx \]  

(5.2)

\[ = \sum_{i,j=0}^{\infty} \sigma_{(i,j)} E(Z_{(i,j)}^r) \]  

(5.3)

where \( Z_{(i,j)}^r \) Denotes the power-parameter Exp-G distribution \( (\alpha + i + j - 1) \).

The mgf of IL-G RV is defined as

\[ M_X(q) = \int_{-\infty}^{\infty} \exp\{qx\} f(x) dx \]  

(5.4)
By expanding Eq. 5.4 using taylor series,

\[ M_X(q) = \sum_{r=0}^{\infty} \frac{q^r}{r!} \int_{-\infty}^{\infty} x^r f(x)dx \]  

(5.5)

Substituting Eq. (5.3) in to the definition of \( M_X(q) \) yields

\[ M_X(q) = \sum_{r=0}^{\infty} \frac{q^r}{r!} E(X^r) \]  

(5.6)

5.2 Order Statistics

Order statistics are used in other areas of statistical theory and procedures to classify outliers in statistical quality control systems. We derive the pdf from the \( p^{th} \) order statistic of the IL-G family of distributions in closed form. Suppose \( X_1, X_2, X_3, X_4 \ldots X_n \) is a random sample from a distribution with pdf \( f(x) \) and let \( X_{1:n}, X_{2:n}, X_{3:n}, X_{4:n} \ldots X_{n:n} \) denotes the corresponding order statistics obtained from this sample. Then

\[ f_{p:n}(x; \theta) = \frac{f(x)}{B_1(p, n - p + 1)} F(x)^{p-1} [1 - F(x)]^{n-p} \]  

(5.7)

where \( f(x) \) and \( F(x) \) are the pdf and CDF of the IL-G distribution as in Eq. (2.2) and Eq. (2.1) respectively. Using the fact that

\[ [1 - F(x)]^{n-p} = \sum_{i=0}^{n-p} (-1)^i \binom{n-p}{i} F(x)^i \]  

(5.8)

By substituting Eq. (5.8) in Eq. (5.7) we have

\[ f_{p:n}(x; \theta) = \sum_{i=0}^{n-p} (-1)^i \binom{n-p}{i} F(x)^i [1 - F(x)]^{n-p} \]  

(5.9)

also,

\[ F(x)^{i+p-1} = \sum_{t=0}^{\infty} \left( -\alpha(i + p - 1) \right)_j \beta^{-\alpha(i+p-1)-j} \left[ \frac{H(x; v)}{H(x; v)} \right]^{-\alpha(i+p-1)-j} \]  

(5.10)

Eqn. (5.9) can be simplified as

\[ f_{p:n}(x; \theta) = \Omega_{ijk} h(x; v) H(x; v)^{\alpha(i+p)+j+k+i-1} \]  

(5.11)

where

\[ \Omega_{ijk} = \frac{\alpha\beta^{-[\alpha(i+p)+k+j]} B_4(p, n - p + 1)}{\alpha \beta^{-[\alpha(i+p)+k+j]}} \sum_{l=0}^{n-p} (-1)^l \binom{n-p}{l} \left( -\alpha(p+i-1) \right)_j \left( 1 + \alpha \right) \frac{\Gamma(\alpha p + k - j + l) + \Gamma(\alpha p + k - j + l - 1)}{\Gamma(\alpha p + k - j + l - 1)} \]  

(5.12)

\( B_1(.) \) is a beta function, and \( h(.) \) also \( H(.) \) are the baseline pdf and cdf respectively.

5.3 Entropy

Here, we consider the Rényi entropy by [16] and Shannon entropy by [17]. One unknown variance measure is called the entropy of a random variable \( X \). The Rényi entropy for IL-G random variable is provided by

\[ I_R(\eta) = \frac{1}{1-\eta} \log \left[ \int_0^\infty f^\eta(x)dx \right] , \eta > 0 \text{and} \eta \neq 1 \]  

(5.13)
where from Eqt. (2.2)
\[
f^\alpha(x) = \left[ \frac{h(x;\alpha,\beta)}{H(x;\alpha,\beta)} \right]^{-\frac{1}{(1+\alpha)}} = \left[ 1 + \frac{\beta H(x;\alpha,\beta)}{H(x;\alpha,\beta)} \right]^{-(1+\alpha)}
\]
Then, by expanding \( f^\alpha(x) \) using a similar process as in Sec. (4) and some simplifications, yields
\[
I_R(\eta) = \frac{1}{(1-\eta)\log} \sum_{i,j=0}^{\infty} \rho(i,j) \int_0^\infty h(x;\nu)^\eta H(x;\nu)^\eta(\alpha-1)+i \, dx
\]
where \( \rho(i,j) = \alpha^i \beta^{-i} (\eta(1+\alpha)^i) \Gamma(\eta(1+\alpha)^i) / \eta(\eta(1+\alpha)^i) \).
Shannon Entropy when \( \eta \uparrow 1 \), is a special case of R\'enyi entropy given by
\[
E \{ -\log [f(x;\nu)] \} = -\log(\alpha\beta) = E \left[ -\log \left( \frac{h(x;\nu)}{H(x;\nu)^2} \right) \right] + (1+\alpha)E \left[ \log \left( 1 + \frac{\beta H(x;\nu)}{H(x;\nu)} \right) \right]
\]
\[
(5.15)
\]
\section{Estimation}
We present the maximum likelihood estimates of the parameters of the IL-G distribution in this section. Let \( x_1, x_2, x_3, \ldots, x_n \) be the observed values of \( n \) observations independently drawn from the ILG family with \( \theta \). The log-likelihood (ll) function for \( \theta \) denoted by \( l(\theta) \) can be expressed as
\[
l(\theta) = n \log(\alpha\beta) + \sum_{i=1}^{n} \log(h(x_i;\nu)) - 2 \sum_{i=1}^{n} \log(H(x_i;\nu)) - (1+\alpha) \sum_{i=1}^{n} \log \left( 1 + \frac{\beta H(x_i;\nu)}{H(x_i;\nu)} \right)
\]
haven taken the partial derivatives of Eqt. (6.1) with respect to \( \alpha, \beta, \) and \( \nu \), we derived \( U(\theta) \) i.e the Score Vector components are as follows
\[
U_\alpha(\theta) = \frac{n}{\alpha} - \sum_{i=1}^{n} \log \left( 1 + \frac{\beta H(x_i;\nu)}{H(x_i;\nu)} \right)
\]
\[
U_\beta(\theta) = \frac{n}{\beta} + \sum_{i=1}^{n} \frac{-1}{1 + \frac{\beta H(x_i;\nu)}{H(x_i;\nu)}}
\]
\[
U_\nu(\theta) = \sum_{i=1}^{n} \left( \frac{h_i(x_i;\nu)}{H(x_i;\nu)} \right) - 2 \sum_{i=1}^{n} \frac{h(x_i;\nu)}{H(x_i;\nu)} + \sum_{i=1}^{n} \frac{(1+\alpha) \beta h(x_i;\nu)}{H^2(x_i;\nu)}
\]
\[
(6.2)
\]
\[
(6.3)
\]
\[
(6.4)
\]
Setting Eqts. (6.2, 6.3, and 6.4) to zero and also solving simultaneously yields the MLE (\( \hat{\theta} \)) of \( \theta \). However, these equations cannot be easily solved analytically. Therefore, statistical software is employed to solve the equations numerically through iterative methods.
\section{Simulation Studies}
A Monte Carlo simulation is conducted and the results of the bias and mean squared error of the various estimated parameter values are presented in Tables 1 and 2. The Monte Carlo simulation is described as follows:
(a) For known parameter values i.e \( \theta = (\alpha, \beta, \lambda, \theta)^T \), samples of different sizes from the Lomax Weibull distribution were generated with small parameter values (\( \alpha = .5, \beta = .8, \lambda = .7, \) and \( \theta = .6 \)) and with relatively big parameter values (\( \alpha = 5, \beta = 10, \lambda = 7, \) and \( \theta = 6 \)) using the quantile function defined in Eqt. (3.3).
Using the maximum likelihood method, we compute the MLE of $\hat{\alpha}_i$, $\hat{\beta}_i$, $\hat{\lambda}_i$, and $\hat{\theta}_i$ for the $i^{th}$ replicate.

Steps (a) and (b) are replicated $N=500$ times.

The bias and MSE for each sample size $n$ are computed as

$$\hat{\theta} = \frac{1}{N} \sum_{i=1}^{N} \hat{\theta}_i,$$

$$\text{Bias}(\hat{\theta}) = (\hat{\theta} - \theta),$$

$$\text{MSE}(\hat{\theta}) = \frac{1}{N} \sum_{i=1}^{N} (\hat{\theta}_i - \theta)^2$$

where $\hat{\theta}_i = (\hat{\alpha}_i, \hat{\beta}_i, \hat{\lambda}_i, \hat{\theta}_i)$ are the mle for the $i^{th}$ replicate. For both small and big parameter values, the sample size $n = 30, 50, 75, 200, 300, 500$ are used to evaluate the behaviour of the bias, variance, and mean squared error as the sample size increases.

It is clear from the simulation study that both bias and mean squared error decrease as the sample size increases for both small and relatively big parameter values. Also, for relatively large sample size, the bias appears very negligible for both small and relatively big parameter values, as shown in Table 1. and Table 2.

**Table 1. Simulation results for the ILW distribution with small parameter values**

<table>
<thead>
<tr>
<th>n</th>
<th>Properties</th>
<th>$\alpha=.5$</th>
<th>$\beta=.7$</th>
<th>$\lambda=.9$</th>
<th>$\theta=.4$</th>
</tr>
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<tr>
<td>30</td>
<td>Bias</td>
<td>0.3533</td>
<td>1.5399</td>
<td>0.5527</td>
<td>0.1136</td>
</tr>
<tr>
<td></td>
<td>MSE</td>
<td>1.1681</td>
<td>9.7608</td>
<td>1.3789</td>
<td>0.2539</td>
</tr>
<tr>
<td></td>
<td>Var.</td>
<td>1.0433</td>
<td>7.4016</td>
<td>1.0734</td>
<td>0.2411</td>
</tr>
<tr>
<td></td>
<td>Est.</td>
<td>0.8533</td>
<td>2.3359</td>
<td>1.2527</td>
<td>0.7136</td>
</tr>
<tr>
<td>50</td>
<td>Bias</td>
<td>0.2538</td>
<td>1.2946</td>
<td>0.5639</td>
<td>0.0135</td>
</tr>
<tr>
<td></td>
<td>MSE</td>
<td>0.5421</td>
<td>7.3381</td>
<td>1.3247</td>
<td>0.1123</td>
</tr>
<tr>
<td></td>
<td>Var.</td>
<td>0.4777</td>
<td>5.6622</td>
<td>1.0067</td>
<td>0.1121</td>
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<tr>
<td></td>
<td>Est.</td>
<td>0.7538</td>
<td>2.0946</td>
<td>1.2639</td>
<td>0.6135</td>
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<tr>
<td>75</td>
<td>Bias</td>
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<td>1.0535</td>
<td>0.4876</td>
<td>-0.0073</td>
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<tr>
<td></td>
<td>MSE</td>
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<td>5.1485</td>
<td>1.0184</td>
<td>0.0836</td>
</tr>
<tr>
<td></td>
<td>Var.</td>
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<td>4.0386</td>
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<td>0.8433</td>
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<tr>
<td></td>
<td>Est.</td>
<td>0.6577</td>
<td>1.8535</td>
<td>1.1876</td>
<td>0.5927</td>
</tr>
<tr>
<td>200</td>
<td>Bias</td>
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<td>0.4979</td>
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<td></td>
<td>MSE</td>
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<td>0.0372</td>
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<td></td>
<td>Var.</td>
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<td>2.2719</td>
<td>0.5612</td>
<td>0.0331</td>
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<tr>
<td></td>
<td>Est.</td>
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<td>1.6796</td>
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<tr>
<td>300</td>
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<td>0.6233</td>
<td>0.3792</td>
<td>-0.0475</td>
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<tr>
<td></td>
<td>MSE</td>
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<td>1.8196</td>
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<td>0.032</td>
</tr>
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<td></td>
<td>Est.</td>
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<td>1.0792</td>
<td>0.5525</td>
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<td>500</td>
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<td>0.3331</td>
<td>-0.0504</td>
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<tr>
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<td>MSE</td>
<td>0.0464</td>
<td>0.9801</td>
<td>0.4249</td>
<td>0.0247</td>
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<td>Var.</td>
<td>0.0375</td>
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<td>0.0222</td>
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<tr>
<td></td>
<td>Est.</td>
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<td>1.2833</td>
<td>1.0331</td>
<td>0.5496</td>
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Table 2. Simulation results for the ILW distribution with big parameter values

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<th>β=10</th>
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<th>θ=6</th>
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<td>-0.6999</td>
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<td>9.9134</td>
</tr>
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<td>50</td>
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<td>-4.9676</td>
<td>-0.8143</td>
<td>3.2122</td>
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<td>47.8544</td>
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<tr>
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<td>19.3307</td>
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<tr>
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<td>6.1857</td>
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<td>13.3521</td>
</tr>
<tr>
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<tr>
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<td>7.6219</td>
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<tr>
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<td>6.3314</td>
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<td>Bias</td>
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<td>-.3457</td>
<td>.6299</td>
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<td>32.4786</td>
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<td>26.4716</td>
<td>.5092</td>
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</tr>
<tr>
<td></td>
<td>Est.</td>
<td>6.5085</td>
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<td>6.6543</td>
<td>6.6299</td>
</tr>
</tbody>
</table>

8 Application

We demonstrate the usefulness of the ILW distribution to the breaking strength of 100 Yarn as reported by [10]. The data-set consists of 63 measurements of the strengths of 1.5 cm glass fibres, which were initially collected by United Kingdom National Physical Laboratory staff as presented below:

0.55, 0.74, 0.77, 0.81, 0.84, 1.24, 0.93, 1.04, 1.11, 1.13, 1.30, 1.25, 1.27, 1.28, 1.29, 1.48, 1.36, 1.39, 1.42, 1.48, 1.51, 1.49, 1.49, 1.50, 1.50, 1.55, 1.52, 1.53, 1.54, 1.55, 1.61, 1.58, 1.59, 1.60, 1.61, 1.63, 1.61, 1.61, 1.62, 1.62, 1.67, 1.64, 1.66, 1.66, 1.66, 1.70, 1.68, 1.68, 1.69, 1.70, 1.73, 1.73, 1.76, 1.76, 1.77, 1.89, 1.81, 1.82, 1.84, 1.84, 2.00, 2.01, 2.24.

We used a maxLik package developed by [18] in R and used by [19]. The goodness-of-fit (gof) statistics used to compare the models’ performance are the Akaike Information Criterion (AIC), AIC with correction for small sample sizes (AICc), and the Bayesian Information Criterion (BIC). Smaller values of the statistics i.e AIC, AICc, and BIC statistics indicates better model fittings. The competing models (distribution) that are used with the ILW are:

(a) The Weibull-Weibull (ww) by [7] with cdf
(b) The Odd Lindley Weibull (OddLW) by [10] with cdf
\[ F_{\text{OddLW}}(x; \alpha, \beta, \lambda) = (1 + \alpha)^{-\alpha} \exp \left\{ -\alpha \left( \frac{\beta}{\beta + (\lambda x)^\beta} \right) \right\} \]
\[ \alpha, \beta, \lambda > 0 \]

(c) The Lomax Weibull (LW) by [8] with cdf
\[ F_{\text{LW}}(x; \alpha, \beta, \lambda, \theta) = 1 - \left( \frac{\beta}{\beta + (\lambda x)^\beta} \right)^\alpha \]
\[ \alpha, \beta, \lambda, \theta > 0 \]

(d) The baseline Weibull (w) with cdf
\[ F_{\text{w}}(x; \lambda, \theta) = 1 - \exp \left\{ -\lambda x^\theta \right\} \]
\[ \lambda, \theta > 0 \]

The competitors are all extensions of the baseline weibull distribution function, with Weibull-Weibull and Lomax Weibull each having four parameters like the new Inverse Lomax Weibull distribution.

The histogram of the Breaking Strength data, and estimated pdf of the ILW and the other competing distributions are presented in Fig. 2. The graph confirms the results of the AIC, AICc, and BIC statistics presented in Table 3 that the ILW distribution fitted the Breaking Strength data better than the other distributions.

Fig. 2. Fitted distributions to Breaking Strength data

As shown in Table (3), the ILW distribution appears to be the best with minimum AIC values, and AICc. Its ranked number 1 outperforming the OddLW by [10] that also used the same data set.
Table 3. MLEs and Standard Errors in parenthesis and -ll, AIC, BIC, AICc for the data set

<table>
<thead>
<tr>
<th>Model</th>
<th>$\alpha$</th>
<th>$\gamma$</th>
<th>$\lambda$</th>
<th>$A$</th>
<th>$-\lambda$</th>
<th>AIC</th>
<th>BIC</th>
<th>AICc</th>
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</thead>
<tbody>
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<td>ILW</td>
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<td>(0.2231)</td>
<td>12.3997</td>
<td>(7.1997)</td>
<td>0.4801</td>
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<td>11.8329</td>
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<td></td>
<td>(0.2231)</td>
<td></td>
<td></td>
<td></td>
<td>(7.1997)</td>
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<tr>
<td>OddILW</td>
<td>0.0491</td>
<td>(0.0871)</td>
<td>1.1021</td>
<td>(0.5271)</td>
<td>0.4921</td>
<td>14.1935</td>
<td>34.3874</td>
<td>34.7938</td>
</tr>
<tr>
<td></td>
<td>(0.0871)</td>
<td></td>
<td></td>
<td></td>
<td>(0.5271)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>W</td>
<td>5.7811</td>
<td>(0.5761)</td>
<td>1.8281</td>
<td>(0.8371)</td>
<td></td>
<td>15.2071</td>
<td>34.1412</td>
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<tr>
<td></td>
<td>(0.5761)</td>
<td></td>
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<td>(0.8371)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>3.2796</td>
<td>(6.1346)</td>
<td>2.4013</td>
<td>(12.3435)</td>
<td>0.1895</td>
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<tr>
<td></td>
<td>(6.1346)</td>
<td></td>
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<td>(12.3435)</td>
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<td></td>
<td></td>
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<tr>
<td>LW</td>
<td>11.9908</td>
<td>(0.0001)</td>
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<td>(0.6532)</td>
<td>0.51082</td>
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<td></td>
<td></td>
<td>(0.6532)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

9 Conclusion

In this paper, we propose and evaluate a new class of distributions called the Inverse Lomax-G (IL-G) Family of Distributions. This family can extend several widely known models. For illustrative purpose, we considered Weibull as base line distribution. Using power series expansion, we derived some of its properties such as an expansion for density function. Some of the derived properties include Moments, Moment generating function, quantile function, order statistics, and entropies. The ILW (by taking Weibull as baseline distribution) parameters were estimated using MLE. The Monte Carlo Simulation results showed that the bias of the MLEs became negligible when sample sizes are greater than 100. The analysis of the Breaking Strength data indicated the superiority of this family of distribution over the competing families.

Competing Interests

Authors have declared that no competing interests exist.

References


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