Backward Doubly SDEs with weak Monotonicity and General Growth Generators

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Authors’ contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

We deal with backward doubly stochastic differential equations (BDSDEs) with a weak monotonicity and general growth generators and a square integrable terminal datum. We show the existence and uniqueness of solutions. As application, we establish the existence and uniqueness of Sobolev solutions to some semilinear stochastic partial differential equations (SPDEs) with a general growth and a weak monotonicity generators. By probabilistic solution, we mean a solution which is representable throughout a BDSDEs.

Keywords: Backward doubly stochastic differential equations; weak monotonicity; Sobolev solutions; semilinear stochastic partial differential equations.

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1 Introduction

Backward doubly stochastic differential equation (BDSDE for short) at the form

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) \, ds + \int_t^T g(s, Y_s, Z_s) \, dB_s - \int_t^T Z_s \, dW_s, \quad 0 \leq t \leq T$$

(E^{\xi, \gamma})

with two different directions of stochastic integrals, i.e., the equation involves both a standard (forward) stochastic integral $dW_t$ and a backward stochastic integral $dB_t$.

The existence and uniqueness of solutions to BDSDEs of type $(E^{\xi, \gamma})$ were first established in Pardoux and Peng [1] they have proved the existence and uniqueness under uniformly Lipschitz conditions. Since then, the BDSDEs have been intensively studied and a lot of papers were devoted to the development of the theory of BDSDEs as well as their relation with the stochastic control problems see [2], [3], [4], and stochastic partial differential equations (SPDEs), we are especially concerned in this paper with the last connection. Was firstly initiated by Pardoux and Peng [1] to give probabilistic interpretation for the solutions of a class of semilinear SPDEs where the coefficients are smooth enough, the idea is to connect the following BDSDEs system

$$Y^{t,x}_s = h(X^{t,x}_T) + \int_s^T f (r, X^{t,x}_r, Y^{t,x}_r, Z^{t,x}_r) \, dr + \int_s^T g (r, X^{t,x}_r, Y^{t,x}_r, Z^{t,x}_r) \, dB_r - \int_s^T Z^{t,x}_r \, dW_r,$n

$$X^{t,x}_s = x + \int_t^s b (X^{t,x}_r) \, dr + \int_t^s \sigma (X^{t,x}_r) \, dW_r,$$

with the following semilinear SPDE

$$u (t, x) = h (x) + \int_s^T \left( \mathcal{L} u (r, x) + f (r, x, u (r, x), \sigma^* \nabla u (r, x)) \right) dr$$

$$+ \int_s^T g (r, x, u (r, x), \sigma^* \nabla u (r, x)) d\bar{B}_r, \quad t \leq s \leq T,$$

where

$$\mathcal{L} := \frac{1}{2} \sum_{i,j} (a_{ij}) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial}{\partial x_i}, \quad \text{with} \quad (a_{ij}) := \sigma \sigma^*.$$n

After what’s realised by Pardoux and Peng [1] numerous authors show the connections between BDSDEs and solutions of stochastic partial differential equations. Bally and Matoussi [5], and [6], [7], studied the solutions of quasilinear SPDEs in Sobolev spaces in terms of BDSDEs with Lipschitz coefficients, in Bahlali et al [8], and [9], they have prove the existence and uniqueness of probabilistic solutions to some semilinear stochastic partial differential equations (SPDEs) with superlinear growth generator, Zhang and Zhao [10] considered BDSDEs under Lipschitz conditions in spatial integral form on infinite horizon and related their solutions with the stationary solutions of certain SPDEs. And then [11], [12], [13] studied the same BDSDE but under linear growth and monotonicity conditions. They also proved that the solution of finite horizon BDSDE gives the solution of the initial value problem of the corresponding SPDE in Sobolev space see [14], and the solution of the infinite horizon BDSDE gives the stationary solution of the SPDEs.

Due to the application of BDSDEs, many works have been made to relax the assumptions on the driver $f$. see for example [8], [15], [16], [17], [18], where Shi et al [16], and [19], [20], provided a comparison theorem which is very important in studying viscosity solution of SPDEs with stochastic tools, and Bahlali et al [8] provided the existence and uniqueness in the case with a superlinear growth generator and a square integrable terminal datum.
In this paper, we obtain existence and uniqueness results for BDSDEs when the coefficient \( f \) has a weak monotonicity and general growth in \( y \) and lipschitz in the variable \( z \), secondly we connect this kind of BDSDEs with the corresponding semilinear SPDEs with superlinear generator for which we establish the existence and uniqueness of Sobolev solutions. The rest of paper is organized as follows.

- In Section 2, we will present some preliminary notations needed in the whole paper.
- In Section 3, we give the estimate for the solutions of BDSDE \( (E_{\xi,f,g}) \).
- In Section 4, we consider our main results, the existence and uniqueness of solution for BDSDE \( (E_{\xi,f,g}) \).
- In Section 5, we give an application to Sobolev solutions for semilinear SPDEs.

## 2 Notations, Assumptions and Definitions

Let \((\Omega, \mathcal{F}, P)\) be a complete probability space. For \( T > 0 \), let \( \{W_t, 0 \leq t \leq T\} \) and \( \{B_t, 0 \leq t \leq T\} \) be two independent standard Brownian motion defined on \((\Omega, \mathcal{F}, P)\) with values in \( \mathbb{R}^d \) and \( \mathbb{R}^l \), respectively.

Let \( \mathcal{F}^W_t := \sigma(W_s; 0 \leq s \leq t) \) and \( \mathcal{F}^B_t := \sigma(B_s - B_t; t \leq s \leq T) \), completed with \( P \)-null sets. We put, \( \mathcal{F}_t := \mathcal{F}^W_t \vee \mathcal{F}^B_t \). It should be noted that \((\mathcal{F}_t)\) is not an increasing family of sub \( \sigma \)– fields, and hence it is not a filtration.

For each \( t \in [0, T] \), we define
\[
\mathcal{G}_t := \mathcal{F}^W_t \vee \mathcal{F}^B_t,
\]
the collection \((\mathcal{G}_t)_{t \in [0, T]}\) is a filtration.

For any \( d, k \geq 1 \), we consider the following spaces of processes:

- Let \( \mathcal{M}^2(0, T; \mathbb{R}^d) \) denote the set of \( d \)-dimensional, \( \mathcal{F}_t \)-measurable stochastic processes \( \{\varphi_t; t \in [0, T]\} \), such that \( E \int_0^T |\varphi_t|^2 dt < \infty \).
- We denote by \( \mathcal{S}^k([0, T], \mathbb{R}^k) \) the set of \( k \)-dimensional continuous, \( \mathcal{F}_t \)-measurable stochastic processes \( \{\varphi_t; t \in [0, T]\} \), which satisfy \( E\left(\sup_{0 \leq t \leq T} |\varphi_t|^2\right) < \infty \).
- \( \mathcal{L}^2 \) the set of \( \mathcal{F}_T \)-measurable random variables \( \xi : \Omega \to \mathbb{R}^k \) with \( E|\xi|^2 < +\infty \).

Let \( f : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \to \mathbb{R}^k \), \( g : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \to \mathbb{R}^{k \times l} \) be measurable functions such that, for every \((y, z) \in \mathbb{R}^k \times \mathbb{R}^{k \times d}\), \( f(\cdot, y, z) \in \mathcal{M}^2(0, T; \mathbb{R}^k) \) and \( g(\cdot, y, z) \in \mathcal{M}^2(0, T, \mathbb{R}^{k \times l}) \).

Now, we consider the following BDSDE
\[
Y_t = \xi + \int_t^T f(s, Y_s, Z_s) \, ds + \int_t^T g(s, Y_s, Z_s) \, dB_s - \int_t^T Z_s \, dW_s, \quad 0 \leq t \leq T \quad (E_{\xi,f,g}),
\]
\( \xi \) is called the terminal datum and \( f \) the generator.

**Definition 2.1.** A solution of equation \((E_{\xi,f,g})\) is a couple \((Y, Z)\) which belongs to the space \( \mathcal{S}^2([0, T], \mathbb{R}^k) \times \mathcal{M}^2([0, T], \mathbb{R}^{k \times d}) \) and satisfies \((E_{\xi,f,g})\).

We consider the following assumptions:

- **(H.1)** \( dP \times dt \text{-a.e. (almost everywhere), } z \in \mathbb{R}^{k \times d}, y \to f(w, t, y, z) \) is continuous.

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Let $\lambda$ with $dP$ 

\[ E \lambda \]

where $dP$ 

\[ f \]

3 Estimate for the solutions of BDSDE 

$\text{BDSE}$ with parameters $(H.2)$ 

\[ f \]

such that 

\[ \exists \gamma \in \mathbb{R}^k, \quad \forall \gamma, z' \in \mathbb{R}^{k \times d}, dP \times dt-a.e., \]

\[ |f(w, t, y, z) - f(w, t, y', z')| \leq c|y - y'| + \alpha|z - z'|. \]

ii) There exists a constant $c > 0$ and a constant $0 < \alpha \leq \frac{1}{2}$ such that $dP \times dt-a.e.,$ 

\[ |g(w, t, y, z) - g(w, t, y', z')| \leq c|y - y'| + \alpha|z - z'|. \]

(H.4) $f$ for $y$ has a general growth, i.e., $dP \times dt-a.e., \forall y \in \mathbb{R}^k$

\[ |f(t, \omega, y, 0)| \leq |f(t, \omega, 0, 0)| + \varphi(|y|), \]

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is increasing continuous function.

(H.5)

\[
\begin{cases}
  f(t, \omega, 0, 0) \in \mathcal{M}^2(0, T, \mathbb{R}^k), \\
  g(t, \omega, 0, 0) \in \mathcal{M}^2(0, T, \mathbb{R}^{k \times d}).
\end{cases}
\]

3 Estimate for the solutions of BDSDE 

$\mathcal{E}^{f,g}$.

We propose the following assumption on $f$ and $g.$

(H.6) $dP \times dt-a.e., \forall (y, z) \in \mathbb{R}^k \times \mathbb{R}^{k \times d}$

\[ \langle y, f(t, \omega, y, z) \rangle \leq \psi(|y|^2) + \lambda|y||z| + |y|\sigma_t, \]

where $\lambda$ is a positive constant, $\sigma_t$ is a positive and $(\mathcal{F}_t)$ progressively measurable processus with $E \int_0^T |\sigma_t|^2 \, dt < \infty$ and $\psi(\cdot)$ is a nondecreasing concave function from $\mathbb{R}^+ \to$ itself with $\psi(0) = 0.$

(H.7) $dP \times dt-a.e., \forall (y, z) \in \mathbb{R}^k \times \mathbb{R}^{k \times d}$

\[ |g(t, \omega, y, z)|^2 \leq \lambda|y|^2 + \gamma|z|^2 + \eta_t, \]

with $\lambda$ is a positive constant, $0 < \gamma \leq \frac{1}{2}$ and $\eta_t$ is a positive and $(\mathcal{F}_t)$ progressively measurable processus with $E \int_0^T \eta_t dt < \infty.$

Proposition 3.1. Let $f$ and $g$ satisfy (H.6) and (H.7), let $(Y_t, Z_t)_{t \in [0, T]}$ be a solution to the BDSDE with parameters $(\xi, T, f, g).$ Then for each $\delta > 0$ there exists a constants $K > 0$ depending only on $\delta, \lambda$ and $\gamma$ such that

(i) for each $0 \leq t \leq T :$

\[
\mathbb{E}(\sup_{0 \leq s \leq T} |Y_s|^2) + \mathbb{E} \left( \int_t^T |Z_s|^2 \, ds \right) \leq \left( \mathbb{E}|\xi|^2 + 2 \int_t^T \psi(\mathbb{E}|Y_s|^2) \, ds + \frac{1}{2} \mathbb{E} \int_t^T |\sigma_s|^2 \, ds \right.
\]

\[ + \left. \mathbb{E} \int_t^T \eta_s \, ds \right) K \exp(K(T - t)). \]
Moreover for each $\delta > 0$ there exists a constants $\bar{K} > 0$ depending only on $\delta, \lambda$ and $\gamma$ such that for $0 \leq r \leq t \leq T$:

\[
\mathbb{E}\left(\sup_{1 \leq s \leq T} |Y_s|^2 | \mathcal{F}_r \right) + \mathbb{E}\left(\int_r^T |Z_s|^2 ds | \mathcal{F}_r \right) \leq \mathbb{E}\left( |\xi|^2 | \mathcal{F}_r \right) + 2 \int_r^T \mathbb{E}\left( |Y_s|^2 | \mathcal{F}_r \right) ds \\
+ \frac{1}{\delta} \left( \int_r^T |\sigma|^2 ds | \mathcal{F}_r \right) + \mathbb{E}\left( \int_r^T \eta_ds | \mathcal{F}_r \right) \mathcal{K}\exp(\bar{K}T).
\]

**Proof:** For the first part, applying It\'s formula to $|Y_t|^2$ yields that, for each $0 \leq t \leq T$,

\[
|Y_t|^2 + \int_t^T |Z_s|^2 ds = |\xi|^2 + 2 \int_t^T (Y_s, f(s, Y_s, Z_s)) ds + 2 \int_t^T \langle Y_s, g(s, Y_s, Z_s) \rangle dB_s \\
- 2 \int_t^T (Y_s, Z_s) dW_s + \int_t^T |g(s, Y_s, Z_s)|^2 ds,
\]

taking expectation, we get

\[
\mathbb{E}|Y_t|^2 + \mathbb{E}\int_t^T |Z_s|^2 ds = \mathbb{E}|\xi|^2 + 2 \mathbb{E}\int_t^T (Y_s, f(s, Y_s, Z_s)) ds + 2 \mathbb{E}\int_t^T \langle Y_s, g(s, Y_s, Z_s) \rangle dB_s \\
- 2 \mathbb{E}\int_t^T (Y_s, Z_s) dW_s + \mathbb{E}\int_t^T |g(s, Y_s, Z_s)|^2 ds.
\]

Now, by (H.6) and Young\'s inequality, we have

\[
2 \int_t^T (Y_s, f(s, Y_s, Z_s)) ds \leq 2 \int_t^T \left( \psi(|Y_s|^2) + \lambda |Y_s| |Z_s| + |Y_s| |\sigma_s| \right) ds, \\
\leq 2 \int_t^T \psi(|Y_s|^2) ds + (2\lambda^2 + \delta) \int_t^T |Y_s|^2 ds \\
+ \int_t^T |\sigma|^2 ds + \int_t^T \frac{|Z_s|^2}{2} ds.
\]

Then by (H.7), we have

\[
\mathbb{E}|Y_t|^2 + \left( \frac{1}{2} - \gamma \right) \mathbb{E}\int_t^T |Z_s|^2 ds \leq \mathbb{E}|\xi|^2 + 2 \mathbb{E}\int_t^T \psi(|Y_s|^2) ds + (2\lambda^2 + \lambda + \delta) \mathbb{E}\int_t^T |Y_s|^2 ds \\
+ \frac{1}{\delta} \mathbb{E}\int_t^T |\sigma|^2 ds + \mathbb{E}\int_t^T \eta_ds.
\]

Since $\int_0^t (Y_s, Z_s) dW_s$ and $\int_0^t (Y_s, g(s, Y_s, Z_s)) dB_s$ are a uniformly integrable martingale. For each $0 \leq t \leq T$, we have the following inequality

\[
\left( \frac{1}{2} - \gamma \right) \mathbb{E}\int_t^T |Z_s|^2 ds \leq \Delta_t,
\]

where

\[
\Delta_t = \mathbb{E}|\xi|^2 + 2 \mathbb{E}\int_t^T \psi(|Y_s|^2) ds + (2\lambda^2 + \lambda + \delta) \mathbb{E}\int_t^T |Y_s|^2 ds + \frac{1}{\delta} \mathbb{E}\int_t^T |\sigma|^2 ds + \mathbb{E}\int_t^T \eta_ds.
\]

Furthermore, it follows from the Burkholder-Davis-Gundy and Young\'s inequality, we have

\[
2 \mathbb{E}\left( \sup_{t \leq s \leq T} \int_u^T \langle Y_s, Z_s \rangle dW_u \right) \leq 2C_p \mathbb{E}\left( \sup_{t \leq s \leq T} |Y_u| \sqrt{\int_t^T |Z_s|^2 ds} \right),
\]

\[
\leq \frac{1 + 2\gamma}{2} \mathbb{E}\left( \sup_{t \leq s \leq T} |Y_u|^2 \right) + \frac{2C_p}{1 + 2\gamma} \mathbb{E}\left( \int_t^T |Z_s|^2 ds \right), \tag{3.2}
\]

\[
< \infty.
\]
By assumptions \((H.6)\), \((H.7)\) and using \((3.1) - (3.2)\), we have
\[
E \left( \sup_{0 \leq s \leq T} |Y_s|^2 \right) + E \int_t^T |Z_s|^2 \, ds \leq \left( \frac{2}{1 - 2\gamma} \right) \left( 1 + \frac{4C_p^2}{(1 + 2\gamma)(1 - 2\gamma)} \right) E (\Delta_t).
\]
Jensen’s inequality, Gronwall’s Lemma and Fubini’s theorem, in view of the concavity condition of \(\psi(\cdot)\), there exists a constant \(K > 0\) such that \(t \in [0, T]\)
\[
E \left( \sup_{0 \leq s \leq T} |Y_s|^2 \right) + E \int_t^T |Z_s|^2 \, ds \leq \left( K E (|g|^2) + 2K \int_t^T \psi(E(Y_s)^2) \, ds \right) + K E \int_t^T |\sigma_s|^2 \, ds + \exp(K(T - t)).
\]
For the second part, we will use the same operation but applied the conditional expectation with respect to \(\mathcal{F}_r\), \(r \in [t, T]\) instead of the mathematical expectation.

Using the Burklhoefer-Davis-Gundy, \(2ab \leq \frac{a^2}{\epsilon} + \epsilon b^2\) inequalities and assumption \((H.7)\), we have
\[
\mathbb{E} \left( \sup_{t \leq s \leq T} \int_t^s (Y_r, g(s, Y_r, Z_r), dB_r) \bigg| \mathcal{F}_r \right) \leq C_p \mathbb{E} \left( \sup_{t \leq s \leq T} |Y_r| \int_t^s |g(s, Y_r, Z_r)|^2 \, ds \bigg| \mathcal{F}_r \right),
\]
\[
\leq \frac{1}{2\epsilon} \mathbb{E} \left( \sup_{t \leq s \leq T} |Y_r|^2 \bigg| \mathcal{F}_r \right) + \frac{\epsilon C_p^2}{2} \mathbb{E} \left( \int_t^s |g(s, Y_r, Z_r)|^2 \, ds \bigg| \mathcal{F}_r \right),
\]
\[
\leq \frac{1}{2\epsilon} \mathbb{E} \left( \sup_{t \leq s \leq T} |Y_r|^2 \bigg| \mathcal{F}_r \right) + \frac{\epsilon C_p^2}{2} \mathbb{E} \left( \int_t^s |Z_s|^2 \, ds \bigg| \mathcal{F}_r \right) + \mathbb{E} \left( \int_t^T \eta_r \, ds \bigg| \mathcal{F}_r \right).
\]
\[(3.3)\]
Applying It’s formula to \(|Y_r|^2, \forall t \in [0, T]\), and using \((H.6), (H.7), (3.3)\) and \(E \left( \int_t^T (Y_r, Z_r) \, dW_r \bigg| \mathcal{F}_r \right) = 0\), we have for any \(0 \leq r \leq t \leq T\)
\[
\mathbb{E} \left( \sup_{t \leq s \leq T} |Y_r|^2 \bigg| \mathcal{F}_r \right) + \mathbb{E} \left( \int_t^T |Z_s|^2 \, ds \bigg| \mathcal{F}_r \right) \leq \mathbb{E} (|\Delta_t|) \mathbb{E} \left( \sup_{t \leq s \leq T} |Y_r|^2 \bigg| \mathcal{F}_r \right) + \frac{1}{2\epsilon} \mathbb{E} \left( \sup_{t \leq s \leq T} |Y_r|^2 \bigg| \mathcal{F}_r \right) + \frac{1}{2} + \left( \frac{2 + \epsilon C_p^2}{2} \right) \mathbb{E} \left( \int_t^T |Z_s|^2 \, ds \bigg| \mathcal{F}_r \right) + \frac{\epsilon C_p^2}{2} \mathbb{E} \left( \int_t^T \eta_r \, ds \bigg| \mathcal{F}_r \right).
\]
Since \(0 \leq \gamma \leq \frac{1}{2}\) it is enough to take \(C_p^2 = \frac{1}{16}\), we have
\[
\mathbb{E} \left( \sup_{t \leq s \leq T} |Y_r|^2 \bigg| \mathcal{F}_r \right) + \mathbb{E} \left( \int_t^T |Z_s|^2 \, ds \bigg| \mathcal{F}_r \right) \leq \mathbb{E} (|\Delta_t|) \mathbb{E} \left( \sup_{t \leq s \leq T} |Y_r|^2 \bigg| \mathcal{F}_r \right) + \frac{1 + \lambda + 1}{2\epsilon} \mathbb{E} \left( \sup_{t \leq s \leq T} |Y_r|^2 \bigg| \mathcal{F}_r \right) + 6\epsilon + \epsilon \mathbb{E} \left( \int_t^T |Z_s|^2 \, ds \bigg| \mathcal{F}_r \right) + 2 \mathbb{E} \left( \int_t^T \eta_r \, ds \bigg| \mathcal{F}_r \right),
\]
we choosing \(\epsilon = \frac{3\lambda + 1}{6}\), get
\[
\mathbb{E} \left( \sup_{t \leq s \leq T} |Y_r|^2 \bigg| \mathcal{F}_r \right) + \mathbb{E} \left( \int_t^T |Z_s|^2 \, ds \bigg| \mathcal{F}_r \right) \leq \mathbb{E} (|\Delta_t|) \mathbb{E} \left( \sup_{t \leq s \leq T} |Y_r|^2 \bigg| \mathcal{F}_r \right) + \frac{3(\lambda + 1)}{4\lambda + 3} \mathbb{E} \left( \sup_{t \leq s \leq T} |Y_r|^2 \bigg| \mathcal{F}_r \right) + \frac{3(\lambda + 1)}{4\lambda + 3} \mathbb{E} \left( \int_t^T |Z_s|^2 \, ds \bigg| \mathcal{F}_r \right) + \frac{3}{4\lambda + 3} \mathbb{E} \left( \int_t^T \eta_r \, ds \bigg| \mathcal{F}_r \right),
\]
since \(0 < \frac{3(\lambda + 1)}{4\lambda + 3} < 1\), we obtain
\[
\mathbb{E} \left( \sup_{t \leq s \leq T} |Y_r|^2 + \int_t^T |Z_s|^2 \, ds \bigg| \mathcal{F}_r \right) \leq \frac{4\lambda + 3}{\lambda} \left( \mathbb{E} (|\Delta_t|) \mathbb{E} \left( \sup_{t \leq s \leq T} |Y_r|^2 \bigg| \mathcal{F}_r \right) + \frac{3}{4\lambda + 3} \mathbb{E} \left( \int_t^T |Z_s|^2 \, ds \bigg| \mathcal{F}_r \right) \right),
\]
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from which together with Gronwall’s Lemma, Fubini’s theorem and Jensen’s inequality, in view of the concavity condition of $\psi(\cdot)$ then there exists a constant $K > 0$ such that for $0 \leq r \leq T$

$$\mathbb{E}\left(\sup_{0 \leq s \leq T} |Y_s|^2 \right) + \mathbb{E}\left(\int_t^T |Z_s|^2 \, ds \right) \leq \mathbb{E}\left(\mathbb{E}\left(\left(\int_0^T |\xi|^2 \, ds\right)^2 \right)|F_t\right) + 2 \int_t^T \psi\left(\mathbb{E}\left(\left|Y_s\right|^2 \right)|F_s\right) \, ds + \frac{1}{2} \mathbb{E}\left(\int_t^T |\sigma_s| \, ds \right) \right)\right)\mathbb{E}\left(\int_t^T \eta_s \, ds \right) \right)K \exp(KT),$$

defines the required result.

### 4 The Main Results

**Theorem 4.1.** Let $\xi \in \mathbb{L}^2$, assume that (H.1)--(H.5) are satisfied. Then equation $(E^F \circ \xi)$ has a unique solution.

#### 4.1 Proof of uniqueness

Suppose that $f$ and $g$ satisfies the assumption (H.1)–(H.5). Let $(Y^1_t, Z^1_t)$ and $(Y^2_t, Z^2_t)$ be two solutions of the BDSDE with parameters $(\xi, T, f, g)$. Then $(Y_t, Z_t) = (Y^1_t - Y^2_t, Z^1_t - Z^2_t)$ is a solution to the following BDSDE

$$\dot{Y}_t = \int_t^T \tilde{f}(s, \bar{Y}_s, \bar{Z}_s) \, ds + \int_t^T \tilde{g}(s, \bar{Y}_s, \bar{Z}_s) \, dB_s - \int_t^T \bar{Z}_s \, dW_s, \quad t \in [0, T],$$

where for each $(\bar{Y}_t, \bar{Z}_t) \in \mathbb{R}^k \times \mathbb{R}^{k \times d}$

$$\left\{\begin{array}{l}
\tilde{f}(t, \bar{Y}_t, \bar{Z}_t) = f(t, \bar{Y}_t + Y^2_t, \bar{Z}_t + Z^2_t) - f(t, Y^2_t, Z^2_t), \\
\tilde{g}(t, \bar{Y}_t, \bar{Z}_t) = g(t, \bar{Y}_t + Y^2_t, \bar{Z}_t + Z^2_t) - g(t, Y^2_t, Z^2_t).
\end{array}\right.$$  

It follows from (H.2) and (H.3) (i) that $dP \times dt - a.e.$,

$$\langle \dot{Y}, \tilde{f}(t, \bar{Y}, Z) \rangle = \langle \dot{Y}, f(t, Y + t, Z) - f(t, Y^2, Z^2) \rangle$$

$$\leq k \left( |\dot{Y}|^2 + c |\bar{Y}|^2 \right).$$

then the generator $\tilde{f}(s, \bar{Y}, \bar{Z})$ of BDSDE with $\psi(u) = k(u), \lambda = c, \sigma_t = 0$ satisfied the assumption (H.6).

It follows from (H.3) (ii) that $dP \times dt - a.e.$,

$$|\tilde{g}(t, \bar{Y}, \bar{Z})|^2 \leq c |\bar{Y}|^2 + \alpha |\bar{Z}|^2,$$

then the assumption (H.7) is satisfied for the generator $\tilde{g}(s, \bar{Y}, \bar{Z})$ of BDSDE with $\alpha = \gamma$ and $\eta_t = 0$.

Thus, it follow from Proposition 3.1 (i) that there exists a constant $K > 0$ depending only on $\delta, \lambda$ and $\gamma$ such that, for each $0 \leq t \leq T$, we have

$$\mathbb{E}\left(\sup_{0 \leq s \leq T} |Y^1_s|^2 \right) + \mathbb{E}\left(\int_t^T |\dot{Z}_s|^2 \, ds \right) \leq C \int_t^T \left( k \left( \mathbb{E}\left(\sup_{0 \leq r \leq T} |\dot{Y}_r|^2 \right) \right) \right) \right)\right)\right)\right)\right)\right)\right)\right),$$

where $C = 2K \exp(2K)$ in view of $\int_0^u k^{-1}(u) \, du = \infty$, Bihari’s inequality yields that, $\forall t \in [0, T]$

$$\mathbb{E}\left(\sup_{0 \leq s \leq T} |Y^1_s|^2 + \int_t^T |\dot{Z}_s|^2 \, ds \right) = 0.$$

The proof of the uniqueness is then complete.
4.2 Proof of existence

Let $\varphi$ be a function of $C^\infty(\mathbb{R}^k,\mathbb{R}^+)$ with the closed unit as compact support, and satisfies $\int_{\mathbb{R}^k} \varphi(v) dv = 1$. For each $n \geq 1$ and each $(\omega, t, Y) \in \Omega \times [0, T] \times \mathbb{R}^k$, we set

\[
\begin{align*}
    f_n(t, Y_t, V_t) &= n^k f(t, Y_t, V_t) \ast \varphi(n Y_t), \\
    &= n^k \int_{\mathbb{R}^k} f(t, v, V_t) \varphi(n (Y_t - v)) dv. 
\end{align*}
\]

(4.1)

Then $f_n$ is an $(\mathcal{F}_t)$-progressively measurable process for each $Y \in \mathbb{R}^k$ and

\[
\begin{align*}
    f_n(t, Y_t, V_t) &= \int_{\mathbb{R}^k} f \left(t, Y_t - \frac{v}{n}, V_t\right) \varphi(v) dv, \\
    &= \int_{\{v:|v|\leq 1\}} f \left(t, Y_t - \frac{v}{n}, V_t\right) \varphi(v) dv. 
\end{align*}
\]

(4.2)

Let us turn to the existence part. The proof will be split into three lemmas and after the proof of Theorem 4.1.

Lemma 4.1. Let $f$ and $g$ satisfies the hypothesis (H.1)-(H.5), $V \in \mathcal{M}^2(0, T, \mathbb{R}^{k \times d})$ and $\xi \in L^2(\mathcal{F}_T, \mathbb{R}^k)$, if there exists a positive constant $\beta$ such that

\[
dP - a.s., |\xi| \leq \beta \quad dP \times dt - a.e. \quad |g(t, \omega, 0, 0)| \leq \beta \quad |f(t, \omega, 0, 0)| \leq \beta \quad |V_t| \leq \beta.
\]

Then there exists a unique solution to the following BDSDE:

\[
Y_t = \xi + \int_t^T f(s, Y_s, V_s) ds + \int_t^T g(s, Y_s, V_s) d\tilde{B}_s - \int_t^T Z_s dW_s, \quad t \in [0, T].
\]

(4.4)

Proof: It follows from (H.3) (i), (H.4) and (4.3) that, for each $Y \in \mathbb{R}^k$, $dP \times dt - a.e.$

\[
|f(s, Y_s, V_s)| \leq c \beta + \beta + \varphi(|Y_s|).
\]

(4.5)

Thus, checked from (4.1) that for each $n \geq 1$, $f_n(t, Y_t, V_t)$ is locally Lipschitz in $Y$ uniformly with respect to $(\omega, t)$. Furthermore, for each $n \geq 1$ and $Y \in \mathbb{R}^k$, it follows from (4.2) and (4.5) that $dP \times dt - a.e.$,

\[
|f_n(t, Y_t, V_t)| = \int_{\{v:|v|\leq 1\}} f(t, Y_t - \frac{v}{n}, V_t) \varphi(v) dv |
\]

\[
\leq (c \beta + \beta + \varphi(|Y_t| + 1)) \int_{\{v:|v|\leq 1\}} \varphi(v) dv = c \beta + \beta + \varphi(|Y_t| + 1).
\]

(4.6)

Now, for some large enough integer $u > 0$ which will be chosen later, let $\rho_u$ be a smooth function such that $0 \leq \rho_u \leq 1$, $\rho_u(Y_t) = 1$ for $|Y_t| \leq u$ and $\rho_u(Y_t) = 0$ as soon as $|Y_t| \geq u + 1$. Then for each $n \geq 1$, the function $\rho_u(Y_t) f_n(t, Y_t, V_t)$ is Lipschitz in $Y$, uniformly with respect to $(t, \omega)$.

Thus, from Pardoux-Peng [15], we know that for each $n \geq 1$, the following BDSDE has a unique solution $(Y^n_t, Z^n_t)_{t \in [0, T]}$:

\[
Y^n_t = \xi + \int_t^T \rho_u(Y^n_s) f_n(s, Y^n_s, V_s) ds + \int_t^T g(s, Y^n_s, V_s) d\tilde{B}_s - \int_t^T Z^n_s dW_s, \quad 0 \leq t \leq T.
\]

(4.7)

It follows from (H.2) and (4.2) that for each $n \geq 1$ and $(Y^n_t, Y^2_t) \in \mathbb{R}^{2k}$, $dP \times dt - a.e.$,

\[
(Y^n_t - Y^2_t, f_n(t, Y^n_t, V) - f_n(t, Y^2_t, V_t)) \leq \int_{\{v:|v|\leq 1\}} k \left(|Y^n_t - Y^2_t|^2\right) \varphi(v) dv = k \left(|Y^n_t - Y^2_t|^2\right).
\]

(4.8)
For each \( n \geq 1 \) and \( Y_t \in \mathbb{R}^k \), combing (4.6) and (4.8) yields that \( dP \times dt - a.e., \)
\[
(Y_t, \rho_u(Y_t) f_n(t, Y_t, V_t)) = \rho_u(Y_t) (Y_t, f_n(t, Y_t, V_t)),
\]
\[
\leq k (|Y_t|^2) + |Y_t| (c \beta + \beta + \phi (1)),
\]
Then the assumption \((H.6)\) is satisfied for the generator \( \rho_u(Y^n_t) f_n(t, Y^n_t, V_t) \) of BDSDE (4.7) with \( \psi(u) = k(u), \lambda = 0, \sigma_t = c \beta + \beta + \phi (1). \)
It follows from \((H.3) (ii)\) that \( dP \times dt - a.e., \)
\[
|g (t, Y^n_t, V_t)|^2 \leq 2|g (t, Y^n_t, V_t) - g (t, 0, 0)|^2 + 2|g (t, 0, 0)|^2,
\]
\[
\leq 2c|Y^n_t|^2 + 2a^2 |V_t|^2 + 2|g (t, 0, 0)|^2.
\]
Thus, it follow from Proposition 3.1 \((ii)\) with \( \delta = 1 \) that there exists a constant \( \bar{K} > 0 \) depending only on \( \delta, \lambda \) and \( \gamma \) such that, for each \( 0 \leq r \leq t \leq T, \) we have
\[
\mathbb{E} \left( |Y^n_t|^2 \right| \mathcal{F}_r) + \mathbb{E} \left( \int_t^T |Z^n_s|^2 ds \right| \mathcal{F}_r) \leq \left( \mathbb{E} \left( |Y^n_0|^2 \right| \mathcal{F}_r) + 2 \int_t^T k \left( \mathbb{E} \left( |Y^n_s|^2 \right| \mathcal{F}_s) \right) ds + (c \beta + \beta + \phi (1))^2 T
\]
\[
+ 4 \mathbb{E} \left( \int_t^T |g(s, \omega, 0)|^2 ds \right| \mathcal{F}_r \right) \mathbb{E} (\bar{K}^T).
\]
Note \( \bar{\theta} = \bar{K} \exp (\bar{K} T) \) and using the (4.3), we get
\[
\mathbb{E} \left( |Y^n_t|^2 \right| \mathcal{F}_r) + \mathbb{E} \left( \int_t^T |Z^n_s|^2 ds \right| \mathcal{F}_r) \leq \bar{\theta}^2 + 2 \bar{\theta} \int_t^T k \left( \mathbb{E} \left( |Y^n_s|^2 \right| \mathcal{F}_s) \right) ds + \bar{\theta} (c \beta + \beta + \phi (1))^2 T + 4 \bar{\theta}^2 T.
\]
Furthermore, since \( k(\cdot) \) is a nondecreasing and concave function with \( k(0) = 0 \) it increases at most linearly, i.e., there exists \( A > 0 \) such that \( k(x) \leq A(x + 1) \) for each \( x \geq 0 \), yields that
\[
\mathbb{E} \left( |Y^n_t|^2 \right| \mathcal{F}_r) + \mathbb{E} \left( \int_t^T |Z^n_s|^2 ds \right| \mathcal{F}_r) \leq \bar{\theta}^2 (A T + 1) + 2 \bar{\theta} A T + \bar{\theta} (c \beta + \beta + \phi (1))^2 T
\]
\[
+ 2 \bar{\theta} A \int_t^T \mathbb{E} \left( |Y^n_s|^2 \right| \mathcal{F}_s) ds.
\]
By Gronwall’s lemma and with \( r = t \), yields that
\[
|Y^n_t|^2 + \mathbb{E} \left( \int_t^T |Z^n_s|^2 ds \right) \leq u^2.
\]
where \( u^2 = (\bar{\theta}^2 (A T + 1) + 2 \bar{\theta} A T + \bar{\theta} (c \beta + \beta + \phi (1))^2 T) \mathbb{E} (2 \bar{\theta} T). \) By the previous inequality, yields that for each \( n \geq 1 \) and \( \forall t \in [0, T] \)
\[
\begin{cases}
|Y^n_t|^2 \leq u^2, \\
\mathbb{E} \left( \int_0^T |Z^n_s|^2 ds \right) \leq u^2.
\end{cases}
\]
(4.9)

By (4.7) and (4.9), we can conclude that \( (Y^n_t, Z^n_t)_{t \in [0, T]} \) solves the following BDSDE:
\[
Y^n_t = \xi + \int_t^T f_n(s, Y^n_s, V_s) ds + \int_t^T g(s, Y^n_s, V_s) d\bar{W}_s - \int_t^T Z^n_s dW_s, \quad 0 \leq t \leq T.
\]
(4.10)

In the sequel, we shall show that \( \left( (Y^n_t, Z^n_t)_{t \in [0, T]} \right)_{n \in \mathbb{N}} \) is a Cauchy sequence in the space \( S^2 (0, T; \mathbb{R}^k) \times M^2 (0, T; \mathbb{R}^{k \times d}). \)
In fact, for each $n \geq 1$ and $m \geq 1$, let $\Delta Y_t^{n,m} = Y_t^m - Y_t^n$ and $\Delta Z_t^{n,m} = Z_t^m - Z_t^n$. Then for each $0 \leq t \leq T$

$$\Delta Y_t^{n,m} = \int_t^T \Delta f^{n,m}(s, \Delta Y_s^{n,m}, V_s) \, ds + \int_t^T \Delta g^{n,m}(s, \Delta Y_s^{n,m}, V_s) \, dB_s - \int_t^T \Delta Z_s^{n,m} \, dW_s,$$  

(4.11)

where

$$\begin{align*}
\Delta f^{n,m}(s, \Delta Y_s^{n,m}, V_s) &= f_n(s, \Delta Y_s^{n,m} + Y_s^m, V_s) - f_m(s, Y_s^m, V_s), \\
\Delta g^{n,m}(s, \Delta Y_s^{n,m}, V_s) &= g(s, \Delta Y_s^{n,m} + Y_s^m, V_s) - g(s, Y_s^m, V_s).
\end{align*}$$

It follows from (4.8) that for each $\Delta Y_t^{n,m} \in \mathbb{R}^k$, $dP \times dt - a.e.,$

$$(\Delta Y_t^{n,m}, \Delta Y_t^{m,n}(t, \Delta Y_t^{n,m}, V_t)) \leq k \left( |\Delta Y_t^{n,m}|^2 + |\Delta Y_t^{m,n}||f_n(t, Y_t^{m}, V_t) - f_m(t, Y_t^{m}, V_t)| \right).$$

Then the the generator $\Delta f^{n,m}(t, \Delta Y_t^{n,m}, V_t)$ of BDSDE (4.11) with $\psi(u) = k(u)$, $\lambda = 0$, $\sigma_1 = |f_n(t, Y_t^{m}, V_t) - f_m(t, Y_t^{m}, V_t)|$ satisfied the assumption (H.6).

It follows from (H.3) (ii) that $dP \times dt - a.e.,$

$$|\Delta g^{n,m}(t, \Delta Y_t^{n,m}, V_t)|^2 \leq c|\Delta Y_t^{n,m}|^2.$$

Then the assumption (H.7) is satisfied for the generator $\Delta g^{n,m}(t, \Delta Y_t^{n,m}, V_t)$ of BDSDE (4.11) with $\lambda = c$, $\gamma = 0$ and $\eta_t = 0$.

Thus, it follow from Proposition 3.1 (i) with $\delta = 1$ that there exists a constant $K > 0$ depending only on $\delta, \lambda$ and $\gamma$ such that, for each $0 \leq t \leq T$

$$E \left( \sup_{0 \leq s \leq T} |\Delta Y_s^{n,m}|^2 \right) + E \left( \int_t^T |\Delta Z_s^{n,m}|^2 \, ds \right) \leq 2\theta \int_t^T \left( E \sup_{s \in [u,T]} |\Delta Y_u^{n,m}|^2 \right) \, ds$$

$$+ \theta E \int_t^T |f_n(s, Y_s^m, V_s) - f_m(s, Y_s^m, V_s)|^2 \, ds,$$

(4.12)

where $\theta = K \exp(K(T-t))$.

On one hand, it follows from (4.2) that, for each $n, m \geq 1$, $t \in [0, T]$ and each $\Delta Y_t^{n,m} \in \mathbb{R}^k$, $dP \times dt - a.e.,$

$$|f_n(t, Y_t^m, V_t) - f_m(t, Y_t^m, V_t)| \leq \int_{\{v: |v| \leq 1\}} \left| f \left( t, Y_t^m - \frac{v}{n}, V_t \right) - f \left( t, Y_t^m - \frac{v}{m}, V_t \right) \right| \phi(v) \, dv,$$

and also from (4.5) and (4.9), we get

$$|f \left( t, Y_t^m - \frac{v}{n}, V_t \right) - f \left( t, Y_t^m - \frac{v}{m}, V_t \right) | \leq 2(\varphi(u+1) + c\beta + \beta) < \infty.$$

Using the continuity of $f$ in $y$, we have

$$\lim_{n,m \to \infty} |f \left( t, Y_t^m - \frac{v}{n}, V_t \right) - f \left( t, Y_t^m - \frac{v}{m}, V_t \right) | = 0,$$

applying Lebesgue’s dominated convergence theorem, we get

$$\lim_{n,m \to \infty} |f_n(t, Y_t^m, V_t) - f_m(t, Y_t^m, V_t) | = 0.$$

On the other hand, we obtain $dP \times dt - a.e.,$

$$|f_n(t, Y_t^m, V_t) - f_m(t, Y_t^m, V_t) | \leq \int_{\{v: |v| \leq 1\}} 2(\varphi(u+1) + c\beta + \beta) \phi(v) \, dv,$$

$$\leq 2(\varphi(u+1) + c\beta + \beta) < \infty,$$
applies again Lebesgue’s dominated convergence theorem, yields that
\[
\lim_{n,m \to \infty} \mathbb{E} \int_t^T |f_n(s, Y^m_s, V_s) - f_m(s, Y^m_s, V_s)|^2 ds = 0. \tag{4.13}
\]

Now, taking the \(\limsup\) in (4.12) and by Fatou’s lemma, monotonicity and continuity of \(k(\cdot)\) and (4.13), we get
\[
\lim_{n,m \to \infty} \sup_{0 \leq t \leq T} \mathbb{E} \left( \sup_{0 \leq s \leq T} |\Delta Y^{n,m}_s|^2 \right) + \mathbb{E} \left( \int_t^T |\Delta Z^{n,m}_s|^2 ds \right) \leq 2B \lim_{n,m \to \infty} \mathbb{E} \left( \sup_{0 \leq s \leq T} |\Delta Y^{n,m}_s|^2 \right) + \mathbb{E} \left( \int_t^T |\Delta Z^{n,m}_s|^2 ds \right).
\]

Thus, in view of \(\int_0^1 k^{-1}(u) \, du = \infty\), Bihari’s inequality yields that, for each \(0 \leq t \leq T\)
\[
\lim_{n,m \to \infty} \mathbb{E} \left( \sup_{0 \leq s \leq T} |\Delta Y^{n,m}_s|^2 \right) + \mathbb{E} \left( \int_t^T |\Delta Z^{n,m}_s|^2 ds \right) = 0,
\]
which means that \((Y^n_t, Z^n_t)_{t \in [0,T]}\) is Cauchy sequence in the space \(\mathcal{S}^2(0,T,\mathbb{R}^k) \times \mathcal{M}^2(0,T,\mathbb{R}^{k \times d})\).

Let \((Y_t, Z_t)_{t \in [0,T]}\) be the limit process of the sequence \((Y^n_t, Z^n_t)_{t \in [0,T]}\) in the process space \(\mathcal{S}^2(0,T,\mathbb{R}^k) \times \mathcal{M}^2(0,T,\mathbb{R}^{k \times d})\).

On one hand, using (4.2), (4.6) and (4.9), we have
\[
|f_n(s, Y^n_s, V_s)| \leq c\beta + \beta + \varphi(|Y^n_s| + 1),
\]
\[
\leq c\beta + \beta + \varphi(u + 1) < \infty,
\]
by definition of \(f_n\) and applying (H.1), we have that \(f_n\) converge a.e. to \(f\). Thus by Lebesgue’s dominated convergence theorem, we have
\[
\lim_{n \to \infty} \int_0^T |f_n(s, Y^n_s, V_s) - f(s, Y_s, V_s)| ds = 0.
\]

In other hand, from the continuity properties of the stochastic integral, it follows that
\[
\sup_{0 \leq t \leq T} \left| \int_t^T g(s, Y^n_s, V_s) \, dB_s - \int_t^T g(s, Y_s, V_s) \, dB_s \right| \to 0,
\]
\[
\sup_{0 \leq t \leq T} \left| \int_t^T Z^n_s \, dW_s - \int_t^T Z_s \, dW_s \right| \to 0, \quad \text{as } n \to \infty \text{ in probability.}
\]

from wich it follow that \(Y^n\) converge uniformly in \(t\) to \(Y\) i.e., \(\lim_{n \to \infty} (\sup_{0 \leq t \leq T} |Y^n_t - Y_t|) = 0\).

Finally, we pass to the limit \(n \to \infty\) in (4.10), we deduce that \((Y_t, Z_t)_{t \in [0,T]}\) solve BDSDE (4.4).

**Lemma 4.2.** Let \(f\) and \(g\) satisfies the hypothesis (H.1)–(H.5), \(V \in \mathcal{M}^2(0,T,\mathbb{R}^{k \times d})\) and \(\xi \in L^2(\mathcal{F}_T,\mathbb{R}^k)\), if there exists a positive constant \(\beta\) such that
\[
dP - \text{a.s.}, \quad |\xi| \leq \beta, \quad dP \times dt - \text{a.e.}, \quad |g(t,\omega, 0,0)| \leq \beta \quad \text{and} \quad |f(t,\omega, 0,0)| \leq \beta. \tag{4.14}
\]

Then there exists a unique solution to the following BDSDE (4.4).

**Proof:** In this lemma, we will eliminate the bounded condition with respecte to the processes \((V_t)_{t \in [0,T]}\) in Lemma 4.1. For each \(n \geq 1\) and \(Z \in \mathbb{R}^{k \times d}\), denote \(q_n(Z) = \frac{Z \cdot X^n}{\sup_{[0,T]} |Z|}\), then \(q_n(Z) = \frac{Z \cdot X^n}{\sup_{[0,T]} |Z|} \leq \inf(|Z|, n)\). It follows from Lemma 4.1, that for each \(n \geq 1\), there exists a solution \((Y^n_t, Z^n_t)_{t \in [0,T]}\) to the following BDSDE
\[
Y^n_t = \xi + \int_t^T f(s, Y^n_s, q_n(V_s)) \, ds + \int_t^T g(s, Y^n_s, q_n(V_s)) \, dB_s - \int_t^T Z^n_s \, dW_s, \quad 0 \leq t \leq T. \tag{4.15}
\]
In the sequel, we shall show that \( \{(Y^n_t, Z^n_t)_{t \in [0,T]}\}_{n \in \mathbb{N}} \) is a Cauchy sequence in the space \( S^2 (0, T; \mathbb{R}^d) \times \mathcal{M}^2 (0, T; \mathbb{R}^{k \times d}) \).

In fact, for each \( n \geq 1 \) and \( m \geq 1 \), let \( \Delta Y_t^{n,m} = Y_t^n - Y_t^m \) and \( \Delta Z_t^{n,m} = Z_t^n - Z_t^m \). Then for each \( 0 \leq t \leq T \)

\[
\Delta Y_t^{n,m} = \int_t^T \Delta f^{n,m}(s, \Delta Y_s^{n,m}, V_s) \, ds + \int_t^T \Delta g^{n,m}(s, \Delta Y_s^{n,m}, V_s) \, dB_s - \int_t^T \Delta Z_s^{n,m} \, dW_s, \quad (4.16)
\]

where

\[
\begin{align*}
\Delta f^{n,m}(s, \Delta Y_s^{n,m}, V_s) &= f(s, \Delta Y_s^{n,m} + Y_s^m, q_n(V_s)) - f(s, Y_s^m, q_m(V_s)), \\
\Delta g^{n,m}(s, \Delta Y_s^{n,m}, V_s) &= g(s, \Delta Y_s^{n,m} + Y_s^m, q_n(V_s)) - g(s, Y_s^m, q_m(V_s)).
\end{align*}
\]

(H.6) and (H.7) is satisfied for the generator \( \Delta f^{n,m}(t, \Delta Y_t^{n,m}, V_t) \) with \( \psi(u) = k(u), \lambda = 0, \sigma_t = |f(t, Y_m^n, q_n(V_t)) - f(t, Y_m^m, q_m(V_t))| \)

respectively \( \Delta g^{n,m}(t, \Delta Y_t^{n,m}, V_t) \) with \( \gamma = \alpha \) and \( \eta_t = 0 \) of BDSDE (4.16).

Indeed by (H.2), we get

\[
\langle \Delta Y_t^{n,m}, \Delta f^{n,m}(t, \Delta Y_t^{n,m}, V_t) \rangle \leq k \left( |\Delta Y_t^{n,m}|^2 + |\Delta Y_t^{n,m}| |f(t, Y_m^n, q_n(V_t)) - f(t, Y_m^m, q_m(V_t))| \right).
\]

and by (H.3) (ii), we have

\[
|\Delta g^{n,m}(t, \Delta Y_t^{n,m}, V_t)|^2 \leq c |\Delta Y_t^{n,m}|^2 + |q_n(V_t) - q_m(V_t)|^2.
\]

Thus, it follow from Proposition 3.1 (i) with \( \delta = 1 \) that there exists a constant \( K > 0 \) depending only on \( \delta, \lambda \) and \( \gamma \) such that, for each \( 0 \leq t \leq T \)

\[
\mathbb{E} \left( \sup_{0 \leq s \leq T} |\Delta Y_s^{n,m}|^2 \right) + \mathbb{E} \left( \int_t^T |\Delta Z_s^{n,m}|^2 \, ds \right) \leq 2K \int_t^T k(\mathbb{E}|\Delta Y_s^{n,m}|^2) \, ds
\]

\[
+ \kappa \mathbb{E} \int_t^T |f(s, Y_s^m, q_n(V_s)) - f(s, Y_s^m, q_m(V_s))|^2 \, ds \exp(K(T-t))
\]

using (H.3) (i) and \( \theta = K \exp(K(T-t)) \), we get

\[
\mathbb{E} \left( \sup_{0 \leq s \leq T} |\Delta Y_s^{n,m}|^2 \right) + \mathbb{E} \left( \int_t^T |\Delta Z_s^{n,m}|^2 \, ds \right) \leq 2\theta \mathbb{E} \int_t^T k(\mathbb{E}|\Delta Y_s^{n,m}|^2) \, ds + \theta \mathbb{E} \int_t^T |q_n(V_s) - q_m(V_s)|^2 \, ds \quad (4.17)
\]

since \( k(x) \leq A(1 + x) \), we obtain

\[
\mathbb{E} \left( \sup_{0 \leq s \leq T} |\Delta Y_s^{n,m}|^2 \right) + \mathbb{E} \left( \int_t^T |\Delta Z_s^{n,m}|^2 \, ds \right) \leq 2\theta AT + 2\theta A \mathbb{E} \int_t^T k(\mathbb{E}|\Delta Y_s^{n,m}|^2) \, ds
\]

\[
+ \theta \mathbb{E} \int_t^T |q_n(V_s) - q_m(V_s)|^2 \, ds.
\]

By taking the lim sup in (4.17), we have

\[
\limsup_{n,m \to \infty} \mathbb{E} \left( \sup_{0 \leq s \leq T} |\Delta Y_s^{n,m}|^2 \right) + \mathbb{E} \left( \int_t^T |\Delta Z_s^{n,m}|^2 \, ds \right) \leq \limsup_{n,m \to \infty} \left( 2\theta \int_t^T k(\mathbb{E}|\Delta Y_s^{n,m}|^2) \, ds \right.
\]

\[
+ \theta \mathbb{E} \int_t^T |q_n(V_s) - q_m(V_s)|^2 \, ds,
\]

respectively for the right-hand side.
Let a positive constant $\beta$. Thus, in view of \(\limsup\) by continuity and monotonicity of $H$, we have

\[
\limsup_{n,m \to \infty} E \left( \sup_{0 \leq s \leq T} |\Delta Y_{s}^{n,m}|^2 + \int_t^T |\Delta Z_{s}^{n,m}|^2 \, ds \right) \leq 2\theta \int_t^T k \left( \limsup_{n,m \to \infty} E |\Delta Y_{s}^{n,m}|^2 \right) \, ds + \theta c E \int_t^T \sup_{n,m \to \infty} |q_n(V_s) - q_m(V_s)|^2 \, ds.
\]

Since

\[
E \int_t^T \sup_{n,m \to \infty} |q_n(V_s) - q_m(V_s)|^2 \, ds = 0.
\]

Thus, in view of \(\int_{0^+} k^{-1}(u) \, du = \infty\), Bihari's inequality yields that, for each \(0 \leq t \leq T\)

\[
\limsup_{n,m \to \infty} E \left( \sup_{0 \leq s \leq T} |\Delta Y_{s}^{n,m}|^2 + \int_t^T |\Delta Z_{s}^{n,m}|^2 \, ds \right) = 0.
\]

We know that \(\left( Y_t^n, Z_t^n \right)_{t \in [0,T]} \) is a Cauchy sequence in the space \(S^2(0,T;\mathbb{R}^k) \times \mathcal{M}^2(0,T;\mathbb{R}^{k \times d})\).

Let \((Y_t, Z_t)_{t \in [0,T]}\) be the limit of the sequence \(\left( Y_t^n, Z_t^n \right)_{t \in [0,T]} \) in the space \(S^2(0,T;\mathbb{R}^k) \times \mathcal{M}^2(0,T;\mathbb{R}^{k \times d})\).

Applying \((H.1), (H.3) (i), (H.4), (4.14)\) and Lebesgue’s dominated convergence theorem, we have

\[
\lim_{n \to \infty} \int_0^T |f(s, Y_s^n, q_n(V_s)) - f(s, Y_s, V_s)| \, ds = 0,
\]

from which it follows that \(Y^n\) converge uniformly in \(t\) to \(Y\), i.e., \(\lim_{n \to \infty} \left( \sup_{0 \leq t \leq T} |Y_t^n - Y_t| \right) = 0\).

Finally, we pass to the limit \(n \to \infty\) in (4.15), we deduce that \((Y_t, Z_t)_{t \in [0,T]} \) solve BDSDE (4.4).

**Lemma 4.3.** Let \(f\) and \(g\) satisfies the hypothesis \((H.1)-(H.5)\) and \(\xi \in L^2(\mathcal{F}_T, \mathbb{R}^k)\), if there exists a positive constant \(\beta\) such that

\[
dP - \text{a.s.}, |\xi| \leq \beta \quad dP \times dt - \text{a.e.}, |g(t, \omega, 0, 0)| \leq \beta \quad \text{and} \quad |f(t, \omega, 0, 0)| \leq \beta. \quad (4.18)
\]

Then there exists a unique solution to the following BDSDE \((E^{\xi,f,g})\).

**Proof:** By Lemma 4.2, we can construct the iterative sequence. Let us set as usual \((Y_t^0, Z_t^0) = (0,0)\) and define recursively, for each \(n \geq 1\)

\[
Y_t^n = \xi + \int_t^T f(s, Y_s^n, Z_s^{n-1}) \, ds + \int_t^T g(s, Y_s^n, Z_s^{n-1}) \, d\tilde{B}_s - \int_t^T Z_s^n \, dW_s, \quad t \in [0,T]. \quad (4.19)
\]

It follows from \((H.2)\) and \((H.3) (i)\) that \(dP \times dt - \text{a.e.}\)

\[
\langle Y_t^n, f(s, Y_s^n, Z_s^{n-1}) \rangle = \langle Y_t^n, f(s, Y_s^n, Z_s^{n-1}) - f(s, 0, Z_s^{n-1}) + f(s, 0, Z_s^{n-1}) \rangle,
\]

\[
\leq k \left( |Y_s^n|^2 + |Y_s^n| \cdot |Z_s^{n-1}| + |f(s, 0, 0)| \right),
\]

then the assumption \((H.6)\) is satisfied for the generator \(f(s, Y_s^n, Z_s^{n-1})\) of BDSDE (4.19) with \(\psi(u) = k(u), \lambda = 0, \sigma_t = c |Z_t^{n-1}| + |f(t, 0, 0)|\).

It follows from \((H.3) (ii)\) that \(dP \times dt - \text{a.e.}\)

\[
|g(t, Y_t^n, Z_t^{n-1})|^2 \leq 2c |Y_t^n|^2 + 2\alpha^2 |Z_t^{n-1}|^2 + 2 |g(t, 0, 0)|^2,
\]

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then the generator \( g(t, Y^n_t, Z^n_{t-1}) \) of BDSDE (4.19) with \( \gamma = 2\alpha^2, \lambda = 2c \) and \( \eta_t = 2 |g(t, 0, 0)|^2 \) satisfied the assumption (H.7).

Thus, it follow from Proposition 3.1 (i) that there exists a constant \( K > 0 \) depending only on \( \delta, \lambda \) and \( \gamma \) such that, for each \( 0 \leq t \leq T \)

\[
\mathbb{E} \left( \sup_{0 \leq s \leq T} |Y^n_s|^2 \right) + \mathbb{E} \left( \int_t^T |Z^n_s|^2 \, ds \right) \leq \left( K \mathbb{E} |\xi|^2 + 2K\int_t^T k \left( \sup_{r \in [s,T]} |Y^n_r|^2 \right) \right) \, ds \\
+ \frac{K}{\delta} \mathbb{E} \int_t^T \left( \frac{1}{\delta} |Y^n_s|^2 + |f(s,0,0)|^2 \right) \, ds \\
+ 2K\mathbb{E} \int_t^T |g(s,0,0)|^2 \, ds \, \exp(K(T-t)).
\]

By \( \theta = K \exp(K(T-t)) \), we note \( H(t) = \theta \left( \mathbb{E} |\xi|^2 + \frac{2}{\delta} \mathbb{E} \int_t^T |f(s,0,0)|^2 \, ds \right) + 2\mathbb{E} \int_t^T |g(s,0,0)|^2 \, ds \). Using (4.18), we have \( H(t) \leq \theta \beta^2 \left( 1 + \frac{2K}{\delta} + 2T \right) = \theta h. \) Therefore

\[
\mathbb{E} \left( \sup_{0 \leq s \leq T} |Y^n_s|^2 \right) + \mathbb{E} \left( \int_t^T |Z^n_s|^2 \, ds \right) \leq \theta h + 2\theta \int_t^T \left( \sup_{r \in [s,T]} |Y^n_r|^2 \right) \, ds + 2\theta K h + \frac{2K}{\delta} \mathbb{E} \int_t^T |Z^n_s|^2 \, ds.
\]

Since \( k(x) \leq A(1 + x) \), we get

\[
\mathbb{E} \left( \sup_{0 \leq s \leq T} |Y^n_s|^2 \right) + \mathbb{E} \left( \int_t^T |Z^n_s|^2 \, ds \right) \leq \theta h + 2\theta h + 2\theta \int_t^T \mathbb{E} \left( \sup_{r \in [s,T]} |Y^n_r|^2 \right) \, ds + 2\theta K h + \frac{2K}{\delta} \mathbb{E} \int_t^T |Z^n_s|^2 \, ds.
\]

Let us set \( \theta h = \max \{ T - \frac{\ln \alpha}{K}, T - \frac{\ln 2}{K\lambda}, 0 \} \). Then for each \( t \in [\theta h, T] \), we have \( H(T-t) \leq 2 \), thus \( \theta \leq 2K \) and

\[
\mathbb{E} \left( \sup_{0 \leq s \leq T} |Y^n_s|^2 \right) + \mathbb{E} \left( \int_t^T |Z^n_s|^2 \, ds \right) \leq 2K + 4KAT + 4AK \int_t^T \mathbb{E} \left( \sup_{r \in [s,T]} |Y^n_r|^2 \right) \, ds + \frac{4K^2}{\delta} \mathbb{E} \int_t^T |Z^n_s|^2 \, ds.
\]

we take \( \delta = 16K^2 \), obtain

\[
\mathbb{E} \left( \sup_{0 \leq s \leq T} |Y^n_s|^2 + \int_t^T |Z^n_s|^2 \, ds \right) \leq 2K + 4KAT + 4AK \int_t^T \mathbb{E} \left( \sup_{r \in [s,T]} |Y^n_r|^2 \right) \, ds + \frac{1}{4} \mathbb{E} \int_t^T |Z^n_s|^2 \, ds.
\]

Applying Growall’s lemma yields that for each \( t \in [\theta h, T] \)

\[
\mathbb{E} \left( \sup_{0 \leq s \leq T} |Y^n_s|^2 + \int_t^T |Z^n_s|^2 \, ds \right) \leq \left( 2K + 4KAT + \frac{1}{4} \mathbb{E} \int_t^T |Z^n_s|^2 \, ds \right) \exp(4AK(T-t)).
\]

For each \( t \in [\theta h, T] \), we have \( \exp(4AK(T-t)) \leq 2 \), then we deduce for each \( n \geq 1 \)

\[
\mathbb{E} \left( \sup_{0 \leq s \leq T} |Y^n_s|^2 + \int_t^T |Z^n_s|^2 \, ds \right) \leq 4K + 8KAT + \frac{1}{2} \mathbb{E} \int_t^T |Z^n_s|^2 \, ds,
\]

\[
\leq 8K + 16KAT. \tag{4.20}
\]

In the sequel, for each \( n \geq 1 \) and \( m \geq 1 \), let \( \Delta Y_{t,n,m}^n = Y_{t,n}^n - Y_{t,m}^m \) and \( \Delta Z_{t,n,m}^n = Z_{t,n}^n - Z_{t,m}^m \). Then

\[
\Delta Y_{t,n,m}^n \sim \int_t^T \Delta f_{t,n,m}^n(s, \Delta Y_{s,n,m}^n) \, ds + \int_t^T \Delta g_{t,n,m}^n(s, \Delta Y_{s,n,m}^n) \, dB_s - \int_t^T \Delta Z_{t,n,m}^n \, dW_s, \tag{4.21}
\]

where

\[
\left\{ \begin{array}{l}
\Delta f_{t,n,m}^n(s, \Delta Y_{s,n,m}^n) = f(s, \Delta Y_{s,n,m}^n + Y_{s,m}^m, Z_{s,m}^{m-1}) - f(s, Y_{s}^m, Z_{s}^{m-1}), \\
\Delta g_{t,n,m}^n(s, \Delta Y_{s,n,m}^n) = g(s, \Delta Y_{s,n,m}^n + Y_{s,m}^m, Z_{s,m}^{m-1}) - g(s, Y_{s}^m, Z_{s}^{m-1}).
\end{array} \right.
\]
It follows from (H.2) and (H.3) that \( dP \times dt - \text{a.e.} \),
\[
\langle \Delta Y^{n,m}_t, \Delta f^{n,m}_t(t, \Delta Y^{n,m}_t) \rangle = \langle \Delta Y^{n,m}_t, f(s, \Delta Y^{n,m}_t + Y^{m}_t, Z^{m-1}_t) - f(s, Y^{m}_t, Z^{m-1}_t) \rangle,
\]
\[
\leq k \left( \langle \Delta Y^{n,m}_t \rangle^2 + |\Delta Y^{n,m}_t| \right) \left| f(t, Y^{m}_t, Z^{m-1}_t) - f(t, Y^{m}_t, Z^{m-1}_t) \right|.
\]
Then the the generator \( \Delta f^{n,m}_t(t, \Delta Y^{n,m}_t) \) of BDSDE (4.21) with
\[
\psi(u) = k(u), \lambda = 0, \sigma_t = |f(t, Y^{m}_t, Z^{m-1}_t) - f(t, Y^{m}_t, Z^{m-1}_t)|.
\]
satisfied the assumption (H.6).

It follows from (H.3) (ii) that \( dP \times dt - \text{a.e.} \),
\[
|\Delta g^{n,m}(s, \Delta Y^{n,m}_t)|^2 \leq c|\Delta Y^{n,m}_t|^2 + \alpha |\Delta Z^{n-1,m-1}_t|.
\]
Then the assumption (H.7) is satisfied for the generator \( \Delta g^{n,m}(t, \Delta Y^{n,m}_t) \) of BDSDE (4.21) with,
\[
c = \gamma \text{ and } \eta_1 = 0.
\]

Thus, it follow from Proposition 3.1 (i) that there exists a constant \( K > 0 \) depending only on \( \delta, \lambda \) and \( \gamma \) such that, for each \( 0 \leq t \leq T \)
\[
\mathbb{E} \left( \operatorname{sup}_{0 \leq s \leq T} |\Delta Y^{n,m}_s|^2 \right) + \mathbb{E} \left( \int_t^T |\Delta z^{n-1,m-1}_t|^2 \right) \leq \exp(K(T-t)) \left( 2K \int_t^T k \operatorname{sup} |\Delta Y^{n,m}_s|^2 \right) ds + \frac{K}{\delta} \int_t^T \left( f(s, Y^{m}_s, Z^{m-1}_s) - f(s, Y^{m}_s, Z^{m-1}_s) \right)^2 ds.
\]
Let us set \( \varrho_1 = \max \{ T - \frac{\ln 2}{\delta}, 0 \} \). Then for each \( t \in [\varrho_1, T] \), we have \( \exp(K(T-t)) \leq 2 \) and
\[
\mathbb{E} \left( \operatorname{sup}_{0 \leq s \leq T} |\Delta Y^{n,m}_s|^2 \right) + \mathbb{E} \left( \int_t^T |\Delta z^{n-1,m-1}_t|^2 \right) \leq 4K \int_t^T k \left( \mathbb{E} |\Delta Y^{n,m}_s|^2 \right) ds + \frac{2Kc^2}{\delta} \int_t^T |\Delta z^{n-1,m-1}_t|^2 ds,
\]
take \( \delta = 8Kc^2 \), we have
\[
\mathbb{E} \left( \operatorname{sup}_{0 \leq s \leq T} |\Delta Y^{n,m}_s|^2 \right) + \mathbb{E} \left( \int_t^T |\Delta z^{n-1,m-1}_t|^2 \right) \leq 4K \int_t^T k \left( \mathbb{E} |\Delta Y^{n,m}_s|^2 \right) ds + \frac{1}{4} \mathbb{E} \left( \int_t^T |\Delta z^{n-1,m-1}_t|^2 \right).
\]
Using monotonicity and continuity of \( k(\cdot) \), (4.20), and taking the limsup in (4.22), by Fatou’s lemma, we have
\[
\limsup_{n,m \to \infty} \left( \mathbb{E} \left( \operatorname{sup}_{0 \leq s \leq T} |\Delta Y^{n,m}_s|^2 \right) + \mathbb{E} \left( \int_t^T |\Delta z^{n-1,m-1}_t|^2 \right) \right) \leq 4K \int_t^T k \left( \limsup_{n,m \to \infty} \mathbb{E} |\Delta Y^{n,m}_s|^2 \right) ds + \frac{1}{4} \mathbb{E} \left( \limsup_{n,m \to \infty} |\Delta z^{n-1,m-1}_t|^2 \right) ds.
\]
Thus, in view of \( \int_0^t k^{-1}(u) du = \infty \), Bihari’s inequality yields that, for each \( \varrho_1 \leq t \leq T \)
\[
\limsup_{n,m \to \infty} \left( \mathbb{E} \left( \operatorname{sup}_{0 \leq s \leq T} |\Delta Y^{n,m}_s|^2 \right) + \mathbb{E} \left( \int_t^T |\Delta z^{n-1,m-1}_t|^2 \right) \right) = 0,
\]
we know that \( \left( Y^{n}_t, Z^n_t \right)_{t \in [\varrho_1, T]} \) is Cauchy sequence in the space \( S^2(\varrho_1, T, \mathbb{R}^k) \times M^2(\varrho_1, T, \mathbb{R}^{k \times d}) \).
Let \( (Y_t, Z_t)_{t \in [\varrho_1, T]} \) be the limit of the sequence \( \left( Y^{n}_t, Z^n_t \right)_{n \in \mathbb{N}} \) in the space \( S^2(\varrho_1, T, \mathbb{R}^k) \times M^2(\varrho_1, T, \mathbb{R}^{k \times d}) \). On the other hand, since \( Z^n_t \) converge in \( M^2(\varrho_1, T, \mathbb{R}^{k \times d}) \) to \( Z_t \), then there exists
a subsequence which will denote $Z^n_t$ such that $\forall n, Z^n_t \to Z_t, dt \otimes dP - a.s.$ and $\sup_n |Z^n_t|$ is $dt \otimes dP$ integrable. Therefore by (H.3) (i) and (H.4), we have
\[ |f(s,Y^n_s, Z^{n-1}_t) - f(s,Y_s, Z_t)| \leq c|Z^{n-1}_t| + |f(s,0,0)| + \varphi(|Y^{n}_s|) < \infty, \]

applying (H.1) and (H.3) (i), we have
\[ \lim_{n \to \infty} |f(s,Y^n_s, Z^{n-1}_t) - f(s,Y_s, Z_t)| = \lim_{n \to \infty} |f(s,Y_s, Z^{n-1}_t) - f(s,Y_s, Z_t)| \leq c \lim_{n \to \infty} |Z^{n-1}_t - Z_t| = 0, \]

thus, $f(s,Y^n_s, Z^{n-1}_t)$ converge a.e. to $f(s,Y_s, Z_t)$. Then by Lebesgue’s dominated convergence theorem, we have
\[ \lim_{n \to \infty} \int_0^T |f(s,Y^n_s, Z^{n-1}_t) - f(s,Y_s, Z_t)| ds = 0. \]

From which it follow that $Y^{n}$ converge uniformly in $t \in [\vartheta_1, T]$ to $Y$ i.e. $\lim_{n \to \infty} (\sup_{t \in [\vartheta_1, T]} |Y^{n}_t - Y_t|) = 0$. Now, we pass to the limit $n \to \infty$ in (4.19), we follow that $(Y_t, Z_t)_{t \in [\vartheta_1]}$ solve BDSDE $(E^{\xi, f, g})$.

Note that $T - \vartheta_1 \geq 0$ and depends only on $c$ and $A$, we can repeat the above operation in finite steps to obtain a solution to the BDSDE $(E^{\xi, f, g})$ on $[\vartheta_2, \vartheta_1]$, $[\vartheta_3, \vartheta_2], \ldots$, and then on $[0, T]$. ■

**Now, proof of Theorem 4.1.** Firstly we approximate $f(t,Y_t, Z_t)$ and $\xi$ by a sequence whose elements satisfy the bound assumption in Lemma 4.3.

For each $n \geq 1$, define $q_n(x) = \frac{q(x)}{\sup_{[0,T]} |q|}$ for each $x \in \mathbb{R}^k$, and let
\[ \xi_n = q_n(\xi) \quad \text{and} \quad f_n(t,Y_t, Z_t) = f(t,Y_t, Z_t) - f(t,0,0) + q_n(f(t,0,0)), \]

clearly, the $f_n$ satisfies (4.18), we have
\[ \lim_{n \to \infty} \mathbb{E}[\xi_n - \xi]^2 = 0 \quad \text{and} \quad \lim_{n \to \infty} \mathbb{E}\left(\int_0^T |q_n(f(s,0,0)) - f(s,0,0)|^2 ds\right) = 0. \]

For each $n \geq 1$, let $(Y^{n}_t, Z^{n}_t)_{t \in [0,T]}$ denote the unique solution to the following BDSDE
\[ Y^{n}_t = \xi_n + \int_0^T f_n(s,Y^n_s, Z^n_s) ds + \int_0^T g(s,Y^n_s, Z^n_s) dB_s - \int_0^T Z^n_s dW_s, \quad 0 \leq t \leq T. \]

In the sequel, for each $n \geq 1$ and $m \geq 1$, let $\Delta Y^{n,m}_t = Y^n_t - Y^m_t$ and $\Delta Z^{n,m}_t = Z^n_t - Z^m_t$. Then
\[ \Delta Y^{n,m}_t = \xi_n - \xi_m + \int_0^T \Delta f^{n,m}(s, Y^{n,m}_s, Z^{n,m}_s) ds + \int_0^T \Delta g^{n,m}(s, Y^{n,m}_s, Z^{n,m}_s) dB_s - \int_0^T \Delta Z^{n,m}_s dB_s, \]

where
\[
\begin{align*}
\Delta f^{n,m}(s, Y^{n,m}_s, Z^{n,m}_s) &= f_n(s, Y^{n,m}_s, Z^{n,m}_s) - f_m(s, Y^m_s, Z^m_s), \\
\Delta g^{n,m}(s, Y^{n,m}_s, Z^{n,m}_s) &= g(s, Y^{n,m}_s, Z^{n,m}_s) - g(s, Y^m_s, Z^m_s).
\end{align*}
\]

By add and subtract, we get
\[
(\Delta Y^{n,m}_s, \Delta f^{n,m}(s, Y^{n,m}_s, Z^{n,m}_s)) = (\Delta Y^{n,m}_s, f_n(s, Y^{n,m}_s, Z^{n,m}_s) - f_m(s, Y^m_s, Z^m_s)) + (\Delta Y^{n,m}_s, f_m(s, Y^{n,m}_s, Z^{n,m}_s) + Y^m_s, Z^m_s - Y^m_s) - f_m(s, Y^{n,m}_s, Z^{n,m}_s) + Z^m_s).
\]

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It follows from (H.2) and (H.3) (i) and (4.23) that $dP \times dt$ a.e.,
\[
(\Delta Y^{n,m}_s, \Delta f^{n,m} (s, \Delta Y^{n,m}_s, \Delta Z^{n,m}_s)) \leq \Delta Y^{n,m}_s, f (s, \Delta Y^{n,m}_s + Y^{n}_s, \Delta Z^{n,m}_s + Z^{n}_s) - f (s, Y^{n}_s, Z^{n}_s) + f (s, Y^{n}_s, Z^{n}_s) + \langle \Delta Y^{n,m}_s, q_n (f (s, 0, 0)) \rangle,
\]
\[
\leq k \left( (\Delta Y^{n,m}_m)^2 + |\Delta Z^{n,m}_m| |\Delta Y^{n,m}_m| + |\Delta Y^{n,m}_m| |q_n (f (s, 0, 0)) - q_m (f (s, 0, 0))| \right).
\]
Then the assumption (H.6) is satisfied for the generator $\Delta f^{n,m} (s, \Delta Y^{n,m}_s, \Delta Z^{n,m}_s)$ of BDSDE (4.26) with $\psi (u) = k (u)$, $\lambda = c$, $\sigma_t = |q_n (f (t, 0, 0)) - q_m (f (t, 0, 0))|$. It follows from (H.3) (ii) that $dP \times dt$ a.e.,
\[
|\Delta g^{n,m} (s, \Delta Y^{n,m}_s, \Delta Z^{n,m}_s)|^2 = |g (s, \Delta Y^{n,m}_s + Y^{n}_s, \Delta Z^{n,m}_s + Z^{n}_s) - g (s, Y^{n}_s, Z^{n}_s)|^2,
\]
\[
\leq c |\Delta Y^{n,m}_m|^2 + \alpha |\Delta Z^{n,m}_m|^2.
\]
Then the generator $\Delta g^{n,m} (s, \Delta Y^{n,m}_s, \Delta Z^{n,m}_s)$ of BDSDE (4.26) with $\lambda = c$, $\alpha = \gamma$ and $\eta_t = 0$ satisfied the assumption (H.7).

Thus, it follows from Proposition 3.1 (i) with $\delta = 1$ that there exists a constant $K > 0$ depending only on $\delta, \lambda$ and $\gamma$ such that, for each $0 \leq t \leq T$
\[
\mathbb{E} \left( \sup_{0 \leq s \leq T} |\Delta Y^{n,m}_s|^2 \right) + \mathbb{E} \left( \int_t^T |\Delta Z^{n,m}_s|^2 ds \right) \leq \theta \mathbb{E} |\xi_n - \xi_m|^2 + 2\theta \int_t^T k \mathbb{E} \left( \sup_{0 \leq r \leq s} |\Delta Y^{n,m}_r|^2 \right) ds
\]
\[
+ \theta \mathbb{E} \int_t^T |q_n (f (s, 0, 0)) - q_m (f (s, 0, 0))|^2 ds.
\]
where $\theta = K \exp (K (T - t))$. Since $k (x) \leq A (1 + x)$, we have
\[
\mathbb{E} \left( \sup_{0 \leq s \leq T} |\Delta Y^{n,m}_s|^2 \right) + \mathbb{E} \left( \int_t^T |\Delta Z^{n,m}_s|^2 ds \right) \leq \theta \mathbb{E} |\xi_n - \xi_m|^2 + 2AT \theta + 2AT \int_t^T \mathbb{E} \left( \sup_{0 \leq r \leq s} |\Delta Y^{n,m}_r|^2 \right) ds
\]
\[
+ \theta \mathbb{E} \int_t^T |q_n (f (s, 0, 0)) - q_m (f (s, 0, 0))|^2 ds.
\]
Using (4.24), we obtain
\[
\mathbb{E} \left( \sup_{0 \leq s \leq T} |\Delta Y^{n,m}_s|^2 \right) + \mathbb{E} \left( \int_t^T |\Delta Z^{n,m}_s|^2 ds \right) \leq 2\theta \mathbb{E} |\xi|^2 + 2AT \theta + 2AT \int_t^T \mathbb{E} \left( \sup_{0 \leq r \leq s} |\Delta Y^{n,m}_r|^2 \right) ds
\]
\[
+ 2\theta \mathbb{E} \int_t^T |f (s, 0, 0)|^2 ds.
\]
Applying Gronwall's lemma yields that for each $t \in [0, T]$ and each $n, m \geq 1$
\[
\mathbb{E} \left( \sup_{0 \leq s \leq T} |\Delta Y^{n,m}_s|^2 \right) + \mathbb{E} \left( \int_t^T |\Delta Z^{n,m}_s|^2 ds \right) \leq \left( 2\theta AT + 2\theta \mathbb{E} |\xi|^2 + 2\theta \mathbb{E} \int_t^T |f (s, 0, 0)|^2 ds \right) \exp (2\theta AT) < \infty.
\]
Taking the lim sup in (4.27) and by previous inequality, Fatou's lemma, monotonicity and continuity of $k (\cdot)$, we have
\[
\limsup_{n, m \to \infty} \mathbb{E} \left( \sup_{0 \leq s \leq T} |\Delta Y^{n,m}_s|^2 + \int_t^T |\Delta Z^{n,m}_s|^2 ds \right) \leq \theta \mathbb{E} \left( \limsup_{n, m \to \infty} |\xi_n - \xi_m|^2 \right)
\]
\[
+ 2\theta \int_t^T k \limsup_{n, m \to \infty} \mathbb{E} \left( \sup_{0 \leq r \leq s} |\Delta Y^{n,m}_r|^2 \right) ds
\]
\[
+ \theta \mathbb{E} \int_t^T \limsup_{n, m \to \infty} |q_n (f (s, 0, 0)) - q_m (f (s, 0, 0))|^2 ds.
\]
\[
= 2\theta \int_t^T k \limsup_{n, m \to \infty} \mathbb{E} \left( \sup_{0 \leq r \leq s} |\Delta Y^{n,m}_r|^2 \right) ds.
\]
Thus, in view of \( \int_0^1 k^{-1} (u) \, du = \infty \), Bihari’s inequality yields that for each \( 0 \leq t \leq T \)
\[
\limsup_{n,m \to \infty} \mathbb{E} \left( \sup_{t \leq s \leq T} |\Delta Y_{s,n,m}^t|^2 + \int_t^T |\Delta Z_{s,n,m}^t|^2 \, ds \right) = 0.
\]

We know that \( \left( (Y^n_t, Z^n_t)_{t \in [0,T]} \right)_{n \in \mathbb{N}^*} \) is Cauchy sequence in the space \( \mathcal{S}^2 \left( (0,T,\mathbb{R}^k) \times \mathcal{M}^2 (0,T,\mathbb{R}^{k \times d}) \right) \). Let \( (Y_t, Z_t)_{t \in [0,T]} \) be the limit of the sequence \( \left( (Y^n_t, Z^n_t)_{t \in [0,T]} \right)_{n \in \mathbb{N}^*} \) in the space \( \mathcal{S}^2 \left( (0,T,\mathbb{R}^k) \times \mathcal{M}^2 (0,T,\mathbb{R}^{k \times d}) \right) \). Using (H.3) (i) and (H.4), we have
\[
|f_n(t, Y^n_t, Z^n_t)| = c|Z^n_t| + |f(t, 0,0)| + \varphi \left( |Y^n_t| \right),
\]
applying (H.1), (H.3) and (4.23), we have \( f_n(s, Y^n_s, Z^n_s) \) converge a.e. to \( f(s, Y_s, Z_s) \). Then by Lebesgue’s dominated convergence theorem, we have
\[
\lim_{n \to \infty} \int_t^T |f_n(s, Y^n_s, Z^n_s) - f(s, Y_s, Z_s)| \, ds = 0.
\]
from which it follow that \( Y^n \) converge uniformly in \( t \) to \( Y \). Now, we pass to the limit \( n \to \infty \) in (4.25), we deduce that \( (Y_t, Z_t)_{t \in [0,T]} \) solve BDSDE \( (E^{f,l,s}) \).

Thus we complete the proof of Theorem 4.1.

5 Application to SPDEs

In this section we connect BDSDEs with weak monotonicity and general growth generators with the correspondent SPDEs and give the Sobolev solution of the SPDEs.

Notation and Definition: \( C^k_b \) set of function of class \( C^k \), whose partial derivatives of order less then or equal to \( k \) are bounded. Given \( x \in \mathbb{R}^d \), \( b \in C^2_b \left( \mathbb{R}^d, \mathbb{R}^d \right) \) and \( \sigma \in C^3_b \left( \mathbb{R}^d, \mathbb{R}^{d \times d} \right) \), denote by \( (X^x_t; \ t \leq s \leq T) \) the unique strong solution of the SDEs following
\[
dX^{t,x}_s = b(X^{t,x}_s) \, ds + \sigma(X^{t,x}_s) \, dW_s, \quad X^{t,x}_t = x. \quad (5.1)
\]
It’s well known that \( \mathbb{E} \left( \sup_{t \leq s \leq T} |X^{t,x}_s|^p \right) < \infty \) for any \( p > 1 \), we recall that the stochastic flow associated to the diffusion processus \( \left( X^{t,x}_s; \ t \leq s \leq T \right) \) is \( (X^{t,x}_s; \ x \in \mathbb{R}^d, \ t \leq s \leq T) \) and the inverse flow is denote by \( \tilde{X}^{t,x}_s \). \( x \to \tilde{X}^{t,x}_s \) is differentiable and we denote by \( J \left( \tilde{X}^{t,x}_s \right) \) the determinant of the Jacobian matrix of \( \tilde{X}^{t,x}_s \), which is positive and satisfies \( J \left( \tilde{X}^{t,x}_s \right) = 1. \)

For \( \phi \in C^\infty_c (\mathbb{R}^d) \) we define the process \( \phi_t : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R} \) by \( \phi_t(s,x) = \phi \left( \tilde{X}^{t,x}_s \right) J \left( \tilde{X}^{t,x}_s \right) \).

Let \( \pi : \mathbb{R}^d \to \mathbb{R}_+ \) be an integrable continues positif function and \( L^2 \left( \mathbb{R}^d, \pi (x) \, dx \right) \) be the weight \( L^2 \) space with weight \( \pi (x) \) endowed with the following norm
\[
||u||_\pi^2 = \int_{\mathbb{R}^d} |u(x)|^2 \pi (x) \, dx.
\]
Let us take the weight \( \pi (x) = \exp \left( F(x) \right) \), where \( F : \mathbb{R}^d \to \mathbb{R} \) is a continues, moreover we assume that there exist some \( R > 0 \) such that \( F \in C^2_{\text{b}} \) for \( |x| > R \), we need the following result of generalized equivalence of norm.
There exist two positive constants $K_1, k_1$ which depend on $T, \pi$, such that for any $t \leq s \leq T$ and $\Phi \in L^1(\Omega \times \mathbb{R}^d, \mathbb{P} \otimes \pi (x) \, dx)$

$$k_1 \left( \int_{\mathbb{R}^d} |\Phi (x)\, \pi (x) \, dx \right) \leq \mathbb{E} \left( \int_{\mathbb{R}^d} |\Phi (X^t)\, \pi (x) \, dx \right) \leq K_1 \left( \int_{\mathbb{R}^d} \Phi (x)\, \pi (x) \, dx \right).$$

Moreover for any $\Psi \in L^1(\Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{P} \otimes dt \otimes \pi (x) \, dx)$

$$k_1 \left( \int_{\mathbb{R}^d} \int_{t}^{T} |\Psi (s, x)\, ds \, \pi (x) \, dx \right) \leq \mathbb{E} \left( \int_{\mathbb{R}^d} \int_{t}^{T} |\Psi (s, X^t)\, ds \, \pi (x) \, dx \right) \leq K_1 \left( \int_{\mathbb{R}^d} \int_{t}^{T} |\Psi (s, x)\, ds \, \pi (x) \, dx \right).$$

Proof: Using the change of variable $y = X^t$, we get

$$\mathbb{E} \left( \int_{\mathbb{R}^d} |\Phi (X^t)\, \pi (x) \, dx \right) = \int_{\mathbb{R}^d} \Phi (y) \mathbb{E} \left( \pi (X^t) \right) dy,$$

and for any $\Psi \in L^1(\Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{P} \otimes dt \otimes \pi (x) \, dx)$

$$\mathbb{E} \left( \int_{\mathbb{R}^d} \int_{t}^{T} |\Psi (s, x)\, ds \, \pi (x) \, dx \right) \leq K_1 \left( \int_{\mathbb{R}^d} \int_{t}^{T} |\Psi (s, x)\, ds \, \pi (x) \, dx \right).$$

By Lemma 5.1 in Bally-Matoussi [5], $k_1 \leq \mathbb{E} \left( \frac{|\Phi (y)||X^t|}{\pi (y)} \right) \leq K_1$ for any $y \in \mathbb{R}^k$, $s \in [t, T]$, the first claim follows. The second claim can be proved similarly. □

Now begin to study the following SPDEs

$$(P^{(f, g)}) \quad \begin{cases} u(t, x) = h(x) + \int_{s}^{T} (L u(r, x) + f(r, x, u(r, x), \sigma^* \nabla u(r, x))) \, dr, \\ + \int_{s}^{T} g(r, x, u(r, x), \sigma^* \nabla u(r, x)) \, d\overline{B}_r, \quad t \leq s \leq T, \end{cases}$$

where

$$L := \frac{1}{2} \sum_{i, j} (a_{ij}) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i} b_i \frac{\partial}{\partial x_i}, \quad \text{with} \quad (a_{ij}) := \sigma \sigma^*.$$

Let $\mathcal{H}$ be the set of random fields $\{u(t, x), 0 \leq t \leq T, x \in \mathbb{R}^d\}$ such that for every $(t, x), u(t, x)$ is $\mathcal{F}^B_{t, t}$-measurable and

$$\|u\|_{\mathcal{H}}^2 = \mathbb{E} \left( \int_{0}^{T} \int_{\mathbb{R}^d} (|u(r, x)|^2 + |(\sigma^* \nabla u(r, x)|^2) \, dr \, dx \right) < \infty.$$ 

The couple $(\mathcal{H}, \| \cdot \|_{\mathcal{H}})$ is a Banach space.

**Definition 5.1.** We say that $u$ is a Sobolev solution to SPDE $(P^{(f, g)})$, if $u \in \mathcal{H}$ and for any $\varphi \in C^1_{c, \infty}([0, T] \times \mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \int_{0}^{T} u(r, x) \frac{\partial \varphi}{\partial x} (r, x) \, dr \, dx + \int_{\mathbb{R}^d} u(r, x) \varphi (r, x) \, dx - \int_{\mathbb{R}^d} h(x) \varphi (T, x) \, dx$$

$$- \frac{1}{2} \int_{\mathbb{R}^d} \int_{0}^{T} \sigma^* u(r, x) \sigma^* \varphi (r, x) \, dr \, dx - \int_{\mathbb{R}^d} \int_{0}^{T} u \sigma \cdot ((b - A) \varphi) (r, x) \, dr \, dx$$

$$= \int_{\mathbb{R}^d} \int_{0}^{T} f(r, x, u(r, x), \sigma^* \nabla u(r, x)) \varphi (r, x) \, dr \, dx + \int_{\mathbb{R}^d} \int_{0}^{T} g(r, x, u(r, x), \sigma^* \nabla u(r, x)) \varphi (r, x) \, d\overline{B}_r \, dx$$

where $A$ is a $d$-vector whose coordinates are defined by $A_j := \frac{1}{2} \sum_{i=1}^{d} \frac{\partial a_{ij}}{\partial x_i}$. 

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In this section we study the Sobolev solution of \( \mathcal{P}^{(f,g)} \) with weak monotonicity and general growth. For \( f : [0,T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \rightarrow \mathbb{R}^k \), \( g : [0,T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \rightarrow \mathbb{R}^{k \times d} \), \( h : \mathbb{R}^d \rightarrow \mathbb{R}^d \).

The main idea is to connect \( \mathcal{P}^{(f,g)} \) with the following BDSDE for each \( s \in [t,T] \)

\[
Y^t,x_s = h(X^t,x_T) + \int_s^T f(r,X^t,x_r,Y^t,x_r,Z^t,x_r) \, dr + \int_s^T g(r,X^t,x_r,Y^t,x_r,Z^t,x_r) \, dB_r - \int_s^T Z^x_r \, dW_r, \quad (5.3)
\]

where \( (X^t,x_s; 0 \leq s \leq T) \) is the solution of SDEs (5.1).

Our object consists to establish the existence and uniqueness of solutions \( u \) to SPDEs \( \mathcal{P}^{(f,g)} \) such that \( u(s, X^t,x_s) = Y^t,x_s \) and \( \sigma \partial_x u(s, X^t,x_s) = Z^x_s \).

We consider the following assumptions (A):

(A.1) For \( (t,x) \) fixed \( dP \times dt \)-a.e., \( x \in \mathbb{R}^d \), \( z \in \mathbb{R}^{k \times d} \) \( y \rightarrow f(w,t,x,y,z) \) is continuous and

\[
\int_{\mathbb{R}^d} \int_0^T |f(t,0,0,0)|^2 \, dt \sigma(x) dx < \infty.
\]

(A.2) \( f \) satisfies the weak monotonicity condition in \( y \), i.e., there exist a nondecreasing and concave function \( k(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) with \( k(u) > 0 \) for \( u > 0 \), \( k(0) = 0 \) and \( \int_0^u k(du) = +\infty \) such that \( dP \times dt \)-a.e., \( \forall y_1, y_2 \in \mathbb{R}^k \), \( z \in \mathbb{R}^{k \times d} \), \( x \in \mathbb{R}^d \)

\[
(y_1 - y_2, f(t,\omega,x,y_1,z) - f(t,\omega,x,y_2,z)) \leq k \left( \sqrt{\sum_{j=1}^d (y_1 - y_2) j} \right).
\]

(A.3) i) \( f \) is Lipschitz in \( z \), uniformly with respect to \((\omega,t,x,y)\) i.e., there exists a constant \( c > 0 \) such that \( dP \times dt \)-a.e.,

\[
|f(\omega,t,x,y,z) - f(\omega,t,x,y,z')| \leq c|z - z'|.
\]

ii) \( \int_{\mathbb{R}^d} \int_0^T |g(t,x,0,0)|^2 \, dt \sigma(x) dx < \infty \) and for \((t,x)\) fixed there exists a constant \( c > 0 \) and a constant \( 0 < \alpha < \frac{1}{4} \) such that \( dP \times dt \)-a.e.,

\[
|g(\omega,t,x,y,z) - g(\omega,t,x,y',z')| \leq c|y - y'| + \alpha|z - z'|.
\]

(A.4) \( f \) have a general growth with respect to \( y \), i.e., \( dP \times dt \)-a.e., \( \forall (x,y) \in \mathbb{R}^d \times \mathbb{R}^k \)

\[
|f(t,\omega,x,y,0)| \leq |f(t,\omega,x,0,0)| + \phi(|y|),
\]

where \( \phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is increasing continuous function.

(A.5) \( h \) belongs to \( L^2(\mathbb{R}^d, \pi(x) \, dx; \mathbb{R}^d) \).

Now by Lemma 5.1, Fubini’s theorem and using (A.1), (A.3)(ii) and (A.5), we have for a.e. \( x \in \mathbb{R}^d \)

\[
\mathbb{E} \left( \int_s^T |f(r,X^t,x_r,0,0)|^2 \, dr + \int_s^T |g(r,X^t,x_r,0,0)|^2 \, dr + |h(X^t,x_s)|^2 \right) < \infty. \quad (5.4)
\]

Hence, it follows from Theorem 4.1, that BDSDEs (5.3) admit a unique solution \( (Y^t,x_s, Z^x_s) \) such that \( Y^t,x_s, Z^x_s \) are \( \mathcal{F}^{W}_{t,s} \cup \mathcal{F}^{B}_{s,t} \) measurable for any \( s \in [0,T] \).
Moreover, by Proposition 3.1 (i) it’s easy to check for each \( \delta > 0 \) that there exists a constant \( K > 0 \) depending only on \( \delta, \lambda \) and \( \gamma \) such that, for each \( 0 \leq t \leq T \)

\[
E \left( \sup_{0 \leq s \leq T} |X_t^s|^2 \right) + \int_t^T |Z_t^s|^2 \, ds \leq \left( \frac{2}{K} h (X_t^s) \right)^2 + 2 \int_t^T k \left( \frac{1}{2} E \left| f \left( s, X_t^s, 0, 0 \right) \right|^2 \right) \, ds + 2E \left( |f \left( s, X_t^s, 0, 0 \right)|^2 \right) \, ds \]

using (5.4) and since \( k (x) \leq A (1 + x) \), we have

\[
E \left( \sup_{0 \leq s \leq T} |Y_t^s|^2 \right) + \int_t^T |Z_t^s|^2 \, ds \leq c + 2\theta A T + 2\theta A \left( \int_t^T E \left( |Y_t^s|^2 \right) \, ds \right),
\]

where \( \theta = K \exp (K (T - T)) \). Finally, applying Gronwall’s lemma, we obtain

\[
E \left( \sup_{0 \leq s \leq T} |Y_t^s|^2 \right) + \int_t^T |Z_t^s|^2 \, ds \leq (c + 2\theta A T) \exp (2\theta A T) < \infty. \tag{5.5}
\]

Now, we are state the main result of this section.

**Theorem 5.1.** Under hypothesis (A), the SPDEs \((\mathcal{P}(f,g))\) admits a unique Sobolev solution \( u \).

Moreover \( u \ (t,x) = Y_t^{t,x} \), where \( (Y_t^{t,x}, Z_t^{t,x})_{t \geq s \leq T} \) is the unique solution of the BDSDEs (5.3) and

\[
u \ (s,X_{s}^{t,x}) = Y_{s}^{t,x} \text{ and } (\sigma^* \nabla u) (s,X_{s}^{t,x}) = Z_{s}^{t,x}, \text{ for a.e. } (s,\omega,x) \text{ in } [t,T] \times \Omega \times \mathbb{R}^d. \tag{5.6}
\]

We first consider the following SPDEs:

\[
\begin{cases}
  u^n \ (t,x) = h (x) + \int_t^T (L u^n (r,x) + f (r, x, u^n (r,x), \sigma^* \nabla u^n (r,x))) \, dr \\
  \quad + \int_t^T g (r, x, u^n (r,x), \sigma^* \nabla u^n (r,x)) \, dB_r, & t \leq s \leq T.
\end{cases}
\]

We need the following results.

**Proposition 5.1.** Under the assumptions (A). Let \( (X_{s}^{t,x}) \) be the unique solution of SDEs (5.1) and for a fixed \( n \in \mathbb{N} \), let \( (Y_{t}^{n,t,x}, Z_{t}^{n,t,x}) \) be the unique solution of the BDSDEs

\[
Y_{s}^{n,t,x} = h (X_{s}^{t,x}) + \int_s^T f (r, X_{s}^{t,x}, Y_{r}^{n,t,x}, Z_{r}^{n,t,x}) \, dr \\
\quad + \int_s^T g (r, X_{s}^{t,x}, Y_{r}^{n,t,x}, Z_{r}^{n,t,x}) \, dB_r - \int_s^T Z_{r}^{n,t,x} \, dW_r. \tag{5.7}
\]

Then for any \( s \in [t,T] \)

\[
Y_{s}^{r,s,X_{s}^{t,x}} = Y_{r}^{r,s,x}, \quad Z_{r}^{r,s,X_{s}^{t,x}} = Z_{r}^{r,s,x}, \text{ for a.e. } r \in [s,T], x \in \mathbb{R}^d.
\]

**Proof:** The proof is similar to the proof of Proposition 3.4 in Q. Zhang and H. Zhao [10].

Using Proposition 3.1, by the same computation as in (5.5), we have that the sequence \( (Y_{s}^{r,s,n}, Z_{s}^{r,s,n}) \) are bounded in \( \mathcal{S}^2 \left( 0, T, \mathbb{R}^d \right) \times \mathcal{M}^2 \left( 0, T, \mathbb{R}^{k \times d} \right) \), i.e.,

\[
\sup_n \left( \sup_{0 \leq s \leq T} |Y_s^{r,s,n}|^2 + \int_t^T |Z_t^{r,s,n}|^2 \, ds \right) < \infty. \tag{5.8}
\]

Also by Proposition 3.1 applying with \( k (\cdot) = \psi (\cdot) \), \( \sigma_t = 0 \) \( \eta_t = 0 \), \( \lambda = c \) and \( \gamma = \alpha \), we can proof by the same computation as in Theorem 4.1, that \( (Y_{s}^{r,s,n}, Z_{s}^{r,s,n})_{s \in [0,T]} \) is a Cauchy sequence in
the space $S^2 (0, T, \mathbb{R}^k) \times M^2 (0, T, \mathbb{R}^{k \times d})$, i.e., there exists a $(Y_{s,t,x}^{n,t,x}, Z_{s,t,x}^{n,t,x})_{s \in [0,T]} \in S^2 (0, T, \mathbb{R}^k) \times M^2 (0, T, \mathbb{R}^{k \times d})$ such that
\[
\mathbb{E} \left( \sup_{0 \leq s \leq T} |Y_{s,t,x}^{n,t,x} - Y_{s,t,x}^{10,t,x}|^2 + \int_t^T |Z_{s,t,x}^{n,t,x} - Z_{s,t,x}^{10,t,x}|^2 \, ds \right) \to 0, \quad \text{as } n \to \infty. \tag{5.9}
\]
Under the assumptions (A) if we define $u^n (t, x) = Y_{t,T,x}^{n,t,x}$ and $\sigma^n \nabla u^n (t, x) = Z_{t,T,x}^{n,t,x}$. Then by a direct application of Proposition 5.1, and Fubini’s Theorem, we have
\[
u^n (s, X_{t,x}^{n,t,x}) = Y_{s,t,x}^{n,t,x}, \quad \sigma^n \nabla u^n (s, X_{t,x}^{n,t,x}) = Z_{s,t,x}^{n,t,x}, \quad \text{for a.e. } s \in [t, T], t \in \mathbb{R}^d. \tag{5.10}
\]

**Theorem 5.2.** Under hypothesis (A), if we define $u^n (s, x) = Y_{s,t,x}^{n,t,x}$. Then the SPDEs $(P(f,g,u^n))$ admits a unique Sobolev solution $u^n$, where $(Y_{n,t,x}^{n,t,x}, Z_{n,t,x}^{n,t,x})_{n \in [0,T]}$ is the unique solution of the BDSDEs (5.7) and
\[
u^n (s, X_{t,x}^{n,t,x}) = Y_{s,t,x}^{n,t,x} \quad \text{and} \quad \sigma^n \nabla u^n (s, X_{t,x}^{n,t,x}) = Z_{s,t,x}^{n,t,x}, \quad \text{for a.e. } s \in [t, T], t \in \mathbb{R}^d. \tag{5.11}
\]

**Proof:** **Existence.** For each $(s, x) \in [t, T] \otimes \mathbb{R}^d$, define $u^n (s, x) = Y_{s,t,x}^{n,t,x}$ and $\sigma^n \nabla u^n (s, x) = Z_{s,t,x}^{n,t,x}$, where $(Y_{n,t,x}^{n,t,x}, Z_{n,t,x}^{n,t,x}) \in \mathcal{S}^2 (0, T, \mathbb{R}^k) \times M^2 (0, T, \mathbb{R}^{k \times d})$ is the solution of Eq (5.7). Then by (5.10)
\[
u^n (s, X_{t,x}^{n,t,x}) = Y_{s,t,x}^{n,t,x}, \quad \sigma^n \nabla u^n (s, X_{t,x}^{n,t,x}) = Z_{s,t,x}^{n,t,x}, \quad \text{for a.e. } s \in [t, T], t \in \mathbb{R}^d.
\]

Set
\[
\left\{ \begin{array}{l}
F^n (s, x) = f (s, x, u^n (s, x), \sigma^n \nabla u^n (s, x)), \\
G^n (s, x) = g (s, x, u^n (s, x), \sigma^n \nabla u^n (s, x)).
\end{array} \right.
\]

Then $(Y_{n,t,x}^{n,t,x}, Z_{n,t,x}^{n,t,x}) \in \mathcal{S}^2 (0, T, \mathbb{R}^k) \times M^2 (0, T, \mathbb{R}^{k \times d})$ solve
\[
Y_{n,t,x}^{n,t,x} = h (X_{t,x}^{n,t,x}) + \int_s^T F^n (r, X_{r,x}^{n,t,x}) \, dr + \int_s^T G^n (r, X_{r,x}^{n,t,x}) \, d\tilde{B}_r - \int_s^T Z_{r,t,x}^{n,t,x} \, dW_r.
\]

Moreover, by Lemma 5.1 and (5.8), we have
\[
\mathbb{E} \left( \int_{t \leq s \leq T} \left( |u^n (s, x)|^2 + |\sigma^n \nabla u^n (s, x)|^2 \right) \, d\pi (x) \right) \leq \frac{1}{\Omega_1} \mathbb{E} \left( \int_t^T \left( |Y_{n,t,x}^{n,t,x}|^2 + |Z_{n,t,x}^{n,t,x}|^2 \right) \, d\pi (x) \right),
\]
\[
\text{From (A.3) (i) and (A.4), we have}
\]
\[
\mathbb{E} \left( \int_{t \leq s \leq T} \left( |F^n (s, x)|^2 + |G^n (s, x)|^2 \right) \, d\pi (x) \right) \leq \frac{1}{\Omega_1} \mathbb{E} \left( \int_t^T \left( |u^n (s, x)|^2 + |\sigma^n \nabla u^n (s, x)|^2 \right) \, d\pi (x) \right).
\]

And from (A.3) (ii), we have
\[
\mathbb{E} \left( \int_{t \leq s \leq T} |G^n (s, x)|^2 \, d\pi (x) \right) \leq \frac{1}{\Omega_1} \mathbb{E} \left( \int_t^T |F^n (s, x)|^2 \, d\pi (x) \right) < \infty.
\]

Using some ideas as in the proof of Theorem 2.1 in [5] similar to the argument as in section 4 in [10], we know that $u^n (t, x)$ is the Sobolev solution of the following SPDE:
\[
\left\{ \begin{array}{l}
u^n (t, x) = h (x) + \int_t^s \left( L^n (u^n (r, x) + F^n (r, x)) \right) dr \\
+ \int_t^s G^n (r, x) \, d\tilde{B}_r, \quad t \leq s \leq T.
\end{array} \right. \tag{5.12}
\]

Noting that by the definition of $F^n (r, x)$ and $G^n (r, x)$, from (5.11), we have that $u^n$ is the Sobolev solution of Eq $(P(f,g,u^n))$. 

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Uniqueness: Let $u^n$ be a solution of Eq. $P^{(f,g,u^n)}$. Define the same notation in the existence part for $F^n$ and $G^n$, since $u^n$ is a solution, so $E \left( \int_{\mathbb{R}^d} \int_t^T \left( |u^n(s,x)|^2 + |\sigma^* \nabla u^n(s,x)|^2 \right) d\sigma \, dx \right) < \infty$. From a similar computation as in existence part, we have

$$E \left( \int_{\mathbb{R}^d} \int_t^T \left( |F^n(s,x)|^2 + |G^n(s,x)|^2 \right) d\sigma \, dx \right) < \infty.$$  

Then, for (5.11) it follows from Proposition 2.3 in [3] that, for and $\phi \in C^c_c(\mathbb{R}^d)$, a.e. $s \in [t,T]$, a.s.

$$\int_{\mathbb{R}^d} \int_t^T u^n(r,x) \phi_t(r,x) \, dr \, dx + \int_{\mathbb{R}^d} u^n(r,x) \phi_t(r,x) \, dr \, dx = \int_{\mathbb{R}^d} h(x) \phi_t(T,x) \, dx - \int_{\mathbb{R}^d} \int_t^T u^n(r,x) \mathcal{L}^* \phi_t(r,x) \, dr \, dx$$

$$= \int_{\mathbb{R}^d} \int_t^T F^n(r,x) \phi_t(r,x) \, dr \, dx + \int_{\mathbb{R}^d} \int_t^T G^n(r,x) \phi_t(r,x) \, dr \, dx.$$

Now using $\phi_t(r,x) = \phi(X_{t,r}^\mu, x)$ and by a change of variable, we get

$$\int_{\mathbb{R}^d} \int_t^T u^n(r,x) \phi_t(r,x) \, dr \, dx = \int_{\mathbb{R}^d} \int_t^T h(x) \phi_t(T,x) \, dx,$$

$$\int_{\mathbb{R}^d} \int_t^T F^n(r,x) \phi_t(r,x) \, dr \, dx = \int_{\mathbb{R}^d} \int_t^T F^n(s,X_{t,r}^\mu) \phi(s) \, ds \, dx,$$

$$\int_{\mathbb{R}^d} \int_t^T G^n(r,x) \phi_t(r,x) \, dr \, dx = \int_{\mathbb{R}^d} \int_t^T G^n(s,X_{t,r}^\mu) \phi(s) \, ds \, dx,$$

by a change of variable $y = X_{t,r}^\mu$ and integration by part formula, we obtain

$$\int_{\mathbb{R}^d} \int_t^T u^n(r,x) \phi_t(r,x) \, dr \, dx = \int_{\mathbb{R}^d} \int_t^T (\sigma^* \nabla u^n) (r,X_{t,r}^\mu) \phi(s) \, ds \, dx + \int_{\mathbb{R}^d} \int_t^T u^n(r,x) \mathcal{L}^* \phi_t(r,x) \, dr \, dx.$$

Hence,

$$\int_{\mathbb{R}^d} \int_t^T u^n(r,X_{t,r}^\mu) \phi(s) \, ds \, dx = \int_{\mathbb{R}^d} h(x) \phi_t(T,x) \, dx + \int_{\mathbb{R}^d} \int_t^T F^n(s,X_{t,r}^\mu) \phi(s) \, ds \, dx$$

$$+ \int_{\mathbb{R}^d} \int_t^T G^n(s,X_{t,r}^\mu) \phi(s) \, ds \, dx - \int_{\mathbb{R}^d} \int_t^T (\sigma^* \nabla u^n) (r,X_{t,r}^\mu) \phi(s) \, ds \, dx.$$

From the arbitrariness of $\phi$ we know that $\left\{ u^n(r,X_{t,r}^\mu), (\sigma^* \nabla u^n)(r,X_{t,r}^\mu), t \leq r \leq T \right\}$ is a solution of the following BDSDE

$$Y_{s,t}^{n,t,x} = h(X_{t}^{\mu,x}) + \int_s^T F^n(r,X_{r,t}^{\mu,x}) \, dr + \int_s^T G^n(r,X_{r,t}^{\mu,x}) \, d\sigma(r), t \leq s \leq T.$$

Then from the definitions of $F^n$ and $G^n$ it follows that $\left\{ u^n(r,X_{t,r}^\mu), (\sigma^* \nabla u^n)(r,X_{t,r}^\mu), t \leq r \leq T \right\}$ solve BDSDE (5.7).

If there is another solution $\tilde{u}^n$ to Eq. $P^{(f,g,u^n)}$, then by the same procedure, we can find another solution $\left(Y_{s,t}^{n,t,x}, \tilde{Z}_{s,t}^{n,t,x} \right)$ solve the BDSDE (5.7), where

$$\tilde{u}^n(s,X_{t,r}^\mu) = Y_{s,t}^{n,t,x}, \quad \sigma^* \nabla \tilde{u}^n(s,X_{t,r}^\mu) = Z_{s,t}^{n,t,x},$$

for a.e. $s \in [t,T], x \in \mathbb{R}^d$.

By Theorem 4.1, the solution of Eq. (5.7) is unique, therefore

$$\tilde{Y}_{s,t}^{n,t,x} = Y_{s,t}^{n,t,x}, \quad \text{for a.e. } s \in [t,T], x \in \mathbb{R}^d.$$

Now, applying Lemma 5.1 again, we have

$$E \left( \int_{\mathbb{R}^d} \int_t^T |\tilde{u}^n(s,x) - u^n(s,x)|^2 \, d\sigma \, dx \right) \leq \frac{1}{k_1} \left( \int_{\mathbb{R}^d} \int_t^T |\tilde{Y}_{s,t}^{n,t,x} - Y_{s,t}^{n,t,x}|^2 \, d\sigma \, dx \right) = 0.$$

So $\tilde{u}^n(s,x) = u^n(s,x)$, for a.e. $s \in [0,T], x \in \mathbb{R}^d$ a.s., uniqueness is proved.
Proposition 5.2. Under assumptions (A), let \( (Y^t_r, Z^t_r) \) be the solution of Eq. (5.3). If we define \( u(s, x) = Y^t_r \), then \( \sigma^* \nabla u(s, x) \) exists for a.e. \( s \in [t, T] \), \( x \in \mathbb{R}^d \) a.s., and
\[
\int_{s_1}^{s_2} \rho^* (s, u(s, x)) \, ds < \infty
\]

Proof: See Proposition 4.2 in Q. Zhang, and H. Zhao [10]

In the rest part of this section, we study Eq \( (P^{(f,g,u^n)}) \). Then by Theorem 5.2, Proposition 5.2, Lemma 5.1 and estimation (5.9), we have
\[
\int_{s}^{T} \rho^* (s, u(s, x)) \, ds < \infty
\]

With (5.14) we prove the Theorem 5.1 in this section.

Proof of Theorem 5.1: Let \( u \in \mathcal{C}^2 \) and \( u^n \) is a Sobolev solution to the problem \( (P^{(f,g,u^n)}) \), then we have
\[
E\left( \int_{0}^{T} \rho^* (s, u(s, x)) \, ds \right) < \infty
\]

By proving that along a subsequence (5.15) converges to (5.2) in \( L^2 (\Omega) \), we have that \( u(t, x) \) satisfies (5.2). We only need to show that along a subsequence as \( n \to \infty \)
\[
\int_{s}^{T} \rho^* (s, u(s, x)) \, ds < \infty
\]

Firstly. Since \( \varphi \in \mathcal{C}^2 \) then \( \varphi \) is belong in \( L^2 (\mathbb{R}^d \times [s, T], dt \otimes dx) \) and by Cauchy-Schwartz inequality, we have
\[
\left| \int_{s}^{T} \rho^* (s, u(s, x)) \, ds \right|^2 < \infty
\]

Also we have by Lemma 5.1, and by definition of \( u^n(r, X^t_r) \), \( \sigma^* \nabla u^n(r, X^t_r) \) that,
\[
\int_{s}^{T} \rho^* (s, u(s, x)) \, ds < \infty
\]

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using (A.3) (i) and \((a + b)^2 \leq 2a^2 + 2b^2\), we have

\[
\mathbb{E} \int_0^\infty \int_s^T |f(r, x, Y_{r}^{s,x}, Z_{r}^{s,x}) - f(r, x, Y_{r}^{s,x}, Z_{r}^{s,x})|^2 d\pi(x) dx \leq 2C \int_0^\infty \int_s^T |Z_{r}^{s,x} - Z_{r}^{s,x}|^2 d\pi(x) dx + 2C \int_0^\infty \int_s^T |f(r, x, Y_{r}^{s,x}, Z_{r}^{s,x}) - f(r, x, Y_{r}^{s,x}, Z_{r}^{s,x})|^2 d\pi(x) dx.
\]

We only need to prove that

\[
\mathbb{E} \int_0^\infty \int_s^T |f(r, x, Y_{r}^{s,x}, Z_{r}^{s,x}) - f(r, x, Y_{r}^{s,x}, Z_{r}^{s,x})|^2 d\pi(x) dx \to 0, \quad \text{as} \ n \to \infty.
\]

Applying assumption (A.1), we have

\[
\lim_{n \to \infty} \mathbb{E} \int_0^\infty \int_s^T |f(r, x, Y_{r}^{s,x}, Z_{r}^{s,x}) - f(r, x, Y_{r}^{s,x}, Z_{r}^{s,x})|^2 d\pi(x) dx = 0.
\]

Since \(\mathbb{E} \int_0^\infty \int_s^T |Z_{r}^{s,x,n}|^2 d\pi(x) dx < \infty\), then there exists a subsequence which we still denote \(Z_{r}^{s,x,n} \to Z_{r}^{s,x}\) such that \(\mathbb{E} \int_0^\infty \int_s^T |Z_{r}^{s,x}|^2 d\pi(x) dx < \infty\), using (5.8), (A.3) (i) and (A.4), we have

\[
\mathbb{E} \int_0^\infty \int_s^T |f(r, x, Y_{r}^{s,x,n}, Z_{r}^{s,x})|^2 d\pi(x) dx \leq \mathbb{E} \int_0^\infty \int_s^T \left(\epsilon |Z_{r}^{s,x}|^2 + |f(r, x, 0, 0)| + \varphi \left(\sup_{s \in [0,T]} |Y_{r}^{s,x}| \right) \right) d\pi(x) dx < \infty.
\]

According to the Lebesgue’s dominated convergence Theorem, it follows that

\[
\mathbb{E} \int_0^\infty \int_s^T |f(r, x, Y_{r}^{s,x,n}, Z_{r}^{s,x}) - f(r, x, Y_{r}^{s,x}, Z_{r}^{s,x})|^2 d\pi(x) dx \to 0, \quad \text{as} \ n \to \infty,
\]

which implies that

\[
\lim_{n \to \infty} \mathbb{E} \int_0^\infty \int_s^T f(r, x, u^n(r, x), \sigma^* \nabla u^n(r, x)) \varphi(r, x) d\pi(x) dx = \mathbb{E} \int_0^\infty \int_s^T f(r, x, u(r, x), \sigma^* \nabla u(r, x)) \varphi(r, x) d\pi(x) dx.
\]

Secondly It remains to prove that

\[
\int_\mathbb{R} d\pi(x) \int_0^\infty \int_s^T g(r, x, u^n(r, x), \sigma^* \nabla u^n(r, x)) \varphi(r, x) d\pi(x) dx,
\]

tends to

\[
\int_\mathbb{R} d\pi(x) \int_0^\infty \int_s^T g(r, x, u(r, x), \sigma^* \nabla u(r, x)) \varphi(r, x) d\pi(x) dx,
\]

as \(n\) tends to \(\infty\). Arguing as in the proof of Theorem 4.1, we get the following limit in probability as \(n \to \infty\),

\[
\int_\mathbb{R} d\pi(x) \int_0^\infty \int_s^T \left(\varphi(r, x) \pi(x) d\mathcal{B}_r \right) \left|X_r^{s,x} - X_r^{s,x} \right| d\pi(x) dx < \infty,
\]

i.e.

\[
\mathbb{E} \int_0^\infty \int_s^T \left(g(r, x, u^n(r, x), \sigma^* \nabla u^n(r, x)) - g(r, x, u(r, x), \sigma^* \nabla u(r, x)) \right) \varphi(r, x) \pi(x) d\mathcal{B}_r \pi^{-1}(x) dx.
\]

Hence, using Lemma 5.1 we get, for every \(s \in [0,T]\)

\[
\int_\mathbb{R} d\pi(x) \int_0^\infty \int_s^T \left(g(r, x, u^n(r, x), \sigma^* \nabla u^n(r, x)) - g(r, x, u(r, x), \sigma^* \nabla u(r, x)) \right) \varphi(r, x) \pi(x) d\mathcal{B}_r \pi^{-1}(x) dx
\]

\[
\leq \frac{1}{4t} \int_\mathbb{R} d\pi(x) \int_0^\infty \int_s^T \left(g(r, x, u^n(r, x), \sigma^* \nabla u^n(r, x)) - g(r, x, u(r, x), \sigma^* \nabla u(r, x)) \right) \varphi(r, x) \pi(x) d\mathcal{B}_r \pi^{-1}(x) dx
\]

\[
= \frac{1}{4t} \int_\mathbb{R} d\pi(x) \int_0^\infty \int_s^T \left(g(r, x, u^n(r, x), Y_{r}^{s,x,n}, Z_{r}^{s,x,n}) - g(r, x, u(r, x), Y_{r}^{s,x}, Z_{r}^{s,x}) \right) \varphi(r, x) \pi(x) d\mathcal{B}_r \pi^{-1}(x) dx.
\]

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Since
\[
\sup_n \mathbb{E} \int_T^s \left( g(r, X_t^n, Y_t^n, Z_t^n) - g(r, X_t^n, Y_t^n, Z_t^n) \right) \varphi(r, X_t^n) \pi(X_t^n) d\mathbb{B}_r < \infty,
\]
and
\[
\int_T^s \left( g(r, X_t^n, Y_t^n, Z_t^n) - g(r, X_t^n, Y_t^n, Z_t^n) \right) \varphi(r, X_t^n) \pi(X_t^n) d\mathbb{B}_r \text{ converges to 0 in probability,}
\]

it follows according to the Lebesgue’s dominated convergence theorem that
\[
\lim_n \mathbb{E} \int_T^s \left( g(r, X_t^n, Y_t^n, Z_t^n) - g(r, X_t^n, Y_t^n, Z_t^n) \right) \varphi(r, X_t^n) \pi(X_t^n) d\mathbb{B}_r = 0.
\]
Therefore \( u(t, x) \) satisfies (5.2), i.e. it is a Sobolev solution of \( (P^{(f, g)}) \). Theorem 5.1 is proved.

6 Conclusion

In this paper we studied the BDSDEs and SPDEs. We introduced a BDSDE with weak monotonicity and general growth generators and a square integrable terminal datum. We studied the relationship between BDSDEs and SPDEs in this case, and we give the Sobolev solutions to some semilinear stochastic partial differential equations (SPDEs) with a general growth and a weak monotonicity generators. By probabilistic solution.

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Competing Interests

Authors have declared that no competing interests exist.

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